UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

Espacios homogéneos infinito-dimensionales

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

Martín Carlos Miglioli

Director de tesis: Gabriel Larotonda
Consejero de estudios: Gabriel Larotonda

Lugar de trabajo: Instituto Argentino de Matemática. "Alberto Calderón". CONICET.

Buenos Aires, 2015

Espacios homogéneos infinito-dimensionales

Esta tesis esta enfocada en aspectos geométricos del análisis funcional relacionados con la gemetría de curvatura negativa de algunos espacios homogéneos modelizados en espacios de Banach. En la primera parte se demuestra en el contexto de estructuras reductivas un teorema de descomposición de Corach-Porta-Recht para espacios simétricos de Finsler de curvatura semi-negativa. Este teorema de descomposición se aplica a la descripción geométrica de complexificaciones de algunos espacios homogéneos de dimensión infinita. En la segunda parte se desarrolla un nuevo enfoque de carácter geométrico a problemas de similaridad. Analizamos en diferentes contextos acciones isométricas naturales en el cono de operadores positivos e inversibles relacionadas con representaciones de grupos y álgebras.

Palabras clave: Álgebras con traza, Grupo de Banach-Lie, Complexificación, Descomposición de Corach-Porta-Recht, Espacio CAT(0), Espacio homogéneo, Estructura de Finsler, Problema de similaridad, Representación acotada, Teorema de punto fijo de Bruhat-Tits, Variedad bandera, Variedad Grassmanniana, Variedad de Stiefel.

Infinite-dimensional homogeneous spaces

This thesis is focused on differential geometric aspects of functional analysis related to the non-positively curved geometry of some homogeneous spaces, which are modeled on Banach spaces. In the first part an extended Corach-Porta-Recht decomposition theorem for Finsler symmetric spaces of semi-negative curvature in the context of reductive structures is proven. This decomposition theorem is applied to give a geometric description of the complexification of some infinite dimensional homogeneous spaces. In the second part a new approach of geometrical nature to similarity problems is developed. We analyze in several contexts a natural isometric action on the cone of positive invertible operators which is related to group and algebra representations.

Keywords: Algebra with trace, Banach-Lie group, Bounded representation, Bruhat-Tits fixed point theorem, CAT(0) space, Coadjoint orbit, Complexification, Corach-Porta-Recht decomposition, Finsler structure, Flag manifold, Grassmann manifold, Operator decomposition, Reductive structure, Stiefel manifold, Similarity problem.

Agradecimientos

En primer lugar a mis padres, Carlos y Elisabeth, y a mis hermanas, Sofía e Inés, por su amor y su apoyo constante a lo largo de estos años.

A mi director de tesis, Gabriel Larotonda, por la constante guia, ayuda y paciencia durante la elaboración de esta tesis.

A todos mis compañeros y amigos del departamento de matemática con los que compartimos charlas y el entusiasmo por la matemática.

A mis amigos de toda la vida, por su incondicionalidad.

A la ANPCyT y al CONICET por las respectivas becas que financiaron este trabajo.

Al Instituto Argentino de Matemática, por darme lugar de trabajo.

A los jurados de esta tesis, por leerla con tanto cuidado.

A lo subcomisión de doctorado por la plena confianza que me tuvo al aceptarme en el doctorado.

A los profesores y docentes auxiliares de los cursos de doctorado y nivelación por el excelente nivel de las materias.

A mis profesores de UdeSA que despertaron el interés en la matemática en mi.

Introducción

Espacios homogéneos de dimensión infinita y curvatura no positiva

En años recientes, el estudio geométrico de álgebras de operadores y sus espacios homogéneos se ha vuelto un tema central en el estudio de la geometría infinito-dimensional. Este estudio es una fuente de ejemplos y contraejemplos, y las técnicas usadas en álgebras de operadores (álgebras de Banach y álgebras C^* con sus herramientas distinguidas) son usadas para obtener resultados sobre variedades de dimensión infinita abstractas, a partir del estudio de sus grupos de automorfismos e isometrias, y del estudio de sus fibrados principales asociados. El lector puede ver el reciente libro [9] de D. Beltiţă para una reseña completa sobre estos objetos y una amplia lista de referencias, véase también la sección "Precedentes" al final de esta Introducción.

Un espacio homogéneo para un grupo de Lie G es una variedad en la que el grupo G actúa transitivamente, i.e. una órbita. Puede ser visto alternativamente como un cociente G/H de un grupo de Banach-Lie G por un subgrupo de Lie H. En el caso en el que el espacio homogéneo es la variedad de operadores positivos e inversibles de un álgebra de operadores (munido de una estructura de Finsler que le da una geometría de curvatura negativa) se puden probar teoremas de descomposición que extienden la usual descomposición polar. Con estos teoremas de descomposición dotamos a las complexificaciones de algunos espacios homogéneos con la estructura de fibrado vectorial asociado, y con estos fibrados vectoriales asociados o fibrados covariantes definimos estructuras complejas adaptadas en los fibrados tangentes de órbitas coadjuntas y órbitas de similaridad unitaria de sistemas de proyecciones (variedades bandera) e isometrias parciales (variedades de Stiefel).

Usando propiedades de la variedad de operadores positivos e inversibles como la convexidad de la distancia a lo largo de geodésicas, la minimalidad de proyecciones sobre subvariedades y la existencia de circuncentros de conjuntos acotados, estudiamos problemas de similaridad desde una perspectiva geométrica. Los problemas de similaridad preguntan en distintos contextos cuándo un grupo H de operadores acotados e inversibles (que actúan en un espacio de Hilbert) es conjugado a un grupo de operadores unitarios. Otras preguntas relacionadas se centran en las propiedades de los operadores positivos e inversibles s tales que $s^{-1}Hs$ es un grupo de operadores unitarios. Si un grupo de operadores inveribles es conjugado a un grupo de operadores unitarios entonces es uniformemente acotado. El enunciado recíproco no vale, por lo que se deben hacer supuestos adicionales sobre el grupo para que éste sea unitarizable. Una variante de este problema es estudiar homomorfismos unitales de álgebras $\pi : A \to \mathcal{B}(\mathcal{H})$, donde A es un álgebra C^*, y estudiar bajo que condiciones sobre el homomorfismo π y el álgebra A, la imagen $\pi(U_A)$ del grupo de unitarios de A es unitarizable. En este caso las órbitas de representaciones son espacios homogéneos con la acción dada por conjugación $g \cdot \pi = g\pi(\cdot)g^{-1}$, donde π es una representación y g es un operador inversible.

Resultados principales

Empezamos haciendo algunas observaciones sobre la notación que va a ser usada. Denotamos variedades con las letras M, N y con las letras x, y, z los puntos de las variedades. Si $f : M \to N$ es un mapa suave entre dos variedades usamos la notación $f_* : TM \to TN$ para el mapa tangente y $f_{*x} : T_xM \to T_{f(x)}N$ para el mapa tangente en $x \in M$. Si $\alpha : I \to M$ es una curva suave entonces definimos como es usual $\dot{\alpha}(t) = \alpha_{*t}(\frac{d}{dt})$. Denotamos a los campos vectoriales con las letras griegas ξ, λ y a los homorfismos con las letras griegas π, ρ. Las letras mayúsculas X, Y, Z denotarán vectores. Los caracteres germánicos $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{p}$ serán usados para denotar álgebras de Lie y sus subespacios. Denotamos con G, H, U a los grupos y con g, h, u, v a sus elementes. Las primeras letras del alfabeto a, b, c serán usadas para denotar operadores positivos e inversibles. Denotamos con V, W y Z espacios de Banach y con U subconjuntos abiertos de estos espacios cuando los consideramos como imágenes de cartas locales.

En el Capítulo 1 introducimos resultados báscos sobre teoría de Lie y sobre espacios simétricos infinito-dimensionales de curvatura negativa que van a ayudar a entender mejor los otros capítulos. Un espacio simétrico de Finsler de curvatura semi-negativa $M = G/U$ se define como un cociente G/U, donde G es un grupo de Banach-Lie, U es el conjunto de puntos fijos de una involución $\sigma : G \to G$ y $\|\cdot\|$ es una norma Ad_U-invariante en $\mathfrak{p} = Ker(\sigma_{*1} + 1) \simeq T_{1U}(G/U)$ que le da a G/U una estructura de Finsler tal que el diferencial en todo punto del mapa exponencial es un operador expansivo. Denotaremos $M = G/U = Sym(G, \sigma, \|\cdot\|)$.

En el Capítulo 2 estudiamos descomposiciones de espacios simétricos de Banach y complexificaciones de espacios homogéneos modelizados en grupos de Banach-Lie. En la Sección 2.2 recordamos la definición de la categoría de pares reductivos introducida por Beltiţă y Galé en [7]. Una estructura reductiva con involución es un cuadruple $(G_A, G_B; E, \sigma)$ tal que:

- G_B es un subgrupo de Lie del grupo de Banach-Lie G_A
- $E : \mathfrak{g}_A \to \mathfrak{g}_B$ es un operador lineal entre las álgebras de Lie de los grupos de Banach-Lie G_A y G_B tal que $Ad_g \circ E = E \circ Ad_g$ para todo $g \in G_B$.
- $\sigma : G_A \to G_A$ es una involución tal que $\sigma(G_B) = G_B$ y $\sigma_{*1} \circ E = E \circ \sigma_{*1}$.

Usando esta categoría y la construcción de un entornorno tubular global obtenida por Conde y Larotonda en [16] obtenemos un teorema de descomposición para sucesiones finitas de pares reductivos de grupos de Banach-Lie:

Teorema. *Si para $n \geq 2$ tenemos las siguientes inclusiones de grupos de Banach-Lie, las siguientes funciones entre sus álgebras de Lie*

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n$$

$$\mathfrak{g}_1 \xleftarrow{E_2} \mathfrak{g}_2 \xleftarrow{E_3} \dots \xleftarrow{E_n} \mathfrak{g}_n$$

y un morfismo $\sigma : G_n \to G_n$ tales que:

- $(G_n, G_{n-1}; E_n, \sigma)$,$(G_{n-1}, G_{n-2}; E_n, \sigma|_{G_{n-1}})$,..., $(G_2, G_1; E_2, \sigma|_{G_2})$ *son estructuras reductivas con involución.*
- $M_n = G_n/U_n = Sym(G_n, \sigma, \|\cdot\|)$ *es un espacio simétrico de Finsler simplemente conexo y de curvatura semi-negativa.*
- $\|E_k|_{\mathfrak{p}_k}\| = 1$ *para $k = 2, \dots, n$, donde usamos la norma del ítem anterior restringida a $\mathfrak{p}_k := \mathfrak{p} \cap \mathfrak{g}_k$.*

Entonces las funciones

$$\Phi_n : U_n \times \mathfrak{p}_{E_n} \times \cdots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1 \to G_n$$

$$(u_n, X_n, \dots, X_2, Y_1) \mapsto u_n e^{X_n} \dots e^{X_2} e^{Y_1}$$

$$\Psi_n : \mathfrak{p}_{E_n} \times \cdots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1 \to G_n^+$$

$$(X_n, \dots, X_2, Y_1) \mapsto e^{Y_1} e^{X_2} \dots e^{X_{n-1}} e^{2X_n} e^{X_{n-1}} \dots e^{X_2} e^{Y_1}$$

son difeomorfismos, donde $\mathfrak{p}_{E_k} := Ker E_k \cap \mathfrak{p}_k$ para $k = 2, \dots, n$.

En la Sección 2.3 la complexificación de algunos espacios homogéneos es estudiada. Si G_B es un subgrupo de un grupo de Banach-Lie G_A y σ es una involución en G_A que deja a G_B invariante, entonces bajo ciertas hipótesis el cociente U_A/U_B de los subgrupos de puntos fijos respectivos de G_A y G_B es una subvariedad $U_A/U_B \hookrightarrow G_A/G_B$ que es el conjunto de puntos fijos de la involución $\sigma_G : G_A/G_B \to G_A/G_B$, $gG_B \mapsto \sigma(g)G_B$. Por lo tanto la variedad compleja G_A/G_B puede ser considerada una complexificación de la variedad real U_A/U_B. El teorema de descomposición es usado para munir a la complexificación G_A/G_B del espacio homogéneo U_A/U_B con la estructura de fibrado vectorial asociado:

Teorema. *Sea $M_A = G_A/U_A = Sym(G_A, \sigma, \|\cdot\|)$ un espacio simétrico de Finsler simplemente conexo y de curvatura semi-negativa, y sea $(G_A, G_B; E, \sigma)$ una estructura reductiva con involución tal que $\|E|_{\mathfrak{p}}\| = 1$. Sea $\Psi_0^E : U_A \times \mathfrak{p}_E \to G_A$, $(u, X) \mapsto ue^X$ y $\kappa : (u, X) \mapsto [(u, X)]$ el mapa cociente. Entonces existe un unico difeomorfismo analítico real y U_A-equivariante $\Psi^E : U_A \times_{U_B} \mathfrak{p}_E \to G_A/G_B$ tal que el diagrama*

$$\begin{array}{ccc} U_A \times \mathfrak{p}_E & \xrightarrow{\Psi_0^E} & G_A \\ \kappa \downarrow & & \downarrow q \\ U_A \times_{U_B} \mathfrak{p}_E & \xrightarrow{\Psi^E} & G_A/G_B \end{array}$$

conmuta, donde $q : G_A \to G_A/G_B$, $g \mapsto gG_B$ es el mapa cociente.

Por lo tanto el espacio homogéneo G_A/G_B tiene la estructura de fibrado vectorial U_A-equivariante sobre U_A/U_B con la proyección dada por la composición

$$G_A/G_B \xrightarrow{(\Psi^E)^{-1}} U_A \times_{U_B} \mathfrak{p}_E \xrightarrow{\quad \Xi \quad} U_A/U_B$$

$$ue^X G_B \mapsto [(u, X)] \mapsto uU_B \qquad \textit{para } u \in U_A \textit{ y } X \in \mathfrak{p}_E$$

y fibra típica $\mathfrak{p}_E$.

Este teorema es usado para construir bajo ciertas hipótesis un isomorfismo $G_A/G_B \simeq T(U_A/U_B)$ entre las complexificaciones y el fibrado tangente de espacios homogéneos de la forma U_A/U_B:

Corolario. *Supongamos las condiciones del teorema anterior y supongamos que G_A es un grupo de Banach-Lie complejo, E es $\mathbb{C}$-lineal y $\mathfrak{u} = i\mathfrak{p}$. Entonces*

$$G_A/G_B \xrightarrow{(\Psi^E)^{-1}} U_A \times_{U_B} \mathfrak{p}_E \xrightarrow{\Theta} U_A \times_{U_B} \mathfrak{u}_E \xrightarrow{\alpha^E} T(U_A/U_B)$$

$$ue^X G_B \mapsto [(u, X)] \mapsto [(u, iX)] \mapsto (\mu_u)_{*o} q_{*1}(iX)$$

es un difeomorfismo U_A-equivariante entre la complexificación G_A/G_B y el fibrado tangente $T(U_A/U_B)$ del espacio homogéneo U_A/U_B. Aquí $\mu_u : U_A/U_B \to U_A/U_B$, $vU_B \mapsto uvU_B$ es una traslación, $\mathfrak{u}$ es el álgebra de Lie de U_A y $\mathfrak{u}_E = KerE \cap \mathfrak{u}$. Con esta identificación la involución $\sigma_G : G_A/G_B \to G_A/G_B$, $gG_B \mapsto \sigma(g)G_B$ es la función $T(U_A/U_B) \to T(U_A/U_B)$, $V \mapsto -V$.

Por lo tanto los fibrados tangentes de una clase de grupos de Banach-Lie pueden ser munidos de una estructura de variedad compleja. En estos casos, el mapa entre sus fibrados tangentes dado por $V \mapsto -V$ es anti-holomorfo como en las estructuras complejas adaptadas estudiadas por Lempert y sus colaboradores, véase [39]. Ejemplos de estos espacios homogéneos son órbitas coadjuntas en ideales de operadores p de Schatten, variedades bandera, y variedades de Stiefel en el contexto de álgebras de operadores, véase [8, 14, 27].

En el Capítulo 3 un nuevo enfoque de orden geométrico a problemas de similaridad es desarrollado. La principal contribución es el análisis en diferentes contextos de la estructura del conjunto de órbitas de la acción isométrica natural de grupos H de elementos inversibles sobre el cono P de elementos positivos inversibles de un álgebra de operadores. Esta acción esta dada por $h \cdot a = hah^*$ con $h \in H$ y $a \in P$.

En la Sección 3.3 la convexidad de la distancia a lo largo de geodésicas en el cono de operadores positivos inversibles es usada para probar la siguiente desigualdad geométrica:

Proposición. *Si $\pi : A \to \mathcal{B}(\mathcal{H})$ es un homorfismo unital acotado entre un álgebra C^* A y el álgebra de operadores acotados que actuan en un espacio de Hilbert $\mathcal{H}$, y s es un operador positivo inversible que minimiza $\|s\|\|s^{-1}\|$ entre los operadores positivos inversibles r tales que $Ad_r \circ \pi = r\pi(\cdot)r^{-1}$ es una $*$-representación, entonces*

$$\|Ad_{s^t} \circ \pi\| \leq \|\pi\|^{1-t} \text{ y } \|Ad_{s^t} \circ \pi\|_{c.b.} = \|\pi\|_{c.b.}^{1-t},$$

donde $\|\cdot\|_{c.b.}$ es la norma completamente acotada de un homomorfismo.

Este resultado fue obtenido por Pisier en [53] usando técnicas de interpolación compleja. Además, propiedades de minimalidad de proyecciones sobre conjuntos convexos en P son usados para probar propiedades de minimalidad de unitarizantes canónicos de homomorfismos unitales $\pi = g\rho(\cdot)g^{-1}$. Aquí g es un operador inversible en $\mathcal{B}(\mathcal{H})$ y $\rho : A \to \mathcal{B}(\mathcal{H})$ es una $*$-representación de un álgebra C^* A tal que existe una esperanza condicional $E : \mathcal{B}(\mathcal{H}) \to \rho(A)'$. Los unitarizantes canónicos fueron obtenidos por Andruchow, Corach y Stojanoff en [2].

Teorema. *Si $\|I - E\| = 1$ entonces el positivo inversible canónico s tal que $Ad_s \circ \pi : A \to \mathcal{B}(\mathcal{H})$ en una $*$-representación satisface $\|s\|\|s^{-1}\| = \|\pi\|_{c.b.}$, i.e. minimiza $\|r\|\|r^{-1}\|$ entre los positivos inversibles r tales que $Ad_r \circ \pi$ es una $*$-representación.*

En la Sección 3.4 estudiamos la existencia de unitarizantes de grupos de operadores inversibles H, i.e. positivos inversibles s tal que sHs^{-1} es un grupo de operadores unitarios, cuando estos grupos actuan en variedades P de operadores positivos inversibles munidos de una métrica derivada a partir de una traza. El teorema de punto fijo de Bruhat-Tits es usado para demostrar que la raiz cuadrada del circuncentro de $\{hh^*\}_{h\in H}$ en P es un unitarizante de H. En el caso de álgebras de von Neumann finitas obtenemos el siguiente resultado de existencia probado con técnicas distintas en [64]:

Teorema. *Si H es un grupo de operadores inversibles en un álgebra de von Neumann finita A tal que $\sup_{h\in H}\|h\| = |H| < \infty$ entonces existe un $s \in \{a \in A : |H|^{-1}1 \leq a \leq |H|1\}$ tal que $s^{-1}Hs$ es un grupo de operadores unitarios en A.*

En este caso mostramos que las subvariedades normales al conjunto de puntos fijos son invariantes bajo la acción $h \cdot a = hah^*$. Si $\mathcal{B}_2(\mathcal{H})$ es el ideal de operadores de Hilbert-Schmidt, entonces probando que la acción canónica de $G = \{g \in \mathcal{B}_2(\mathcal{H}) + \mathbb{C}1 : g \text{ es inversible}\}$ sobre $P = \{g \in \mathcal{B}_2(\mathcal{H}) + \mathbb{C}1 : g > 0\}$ restringida a algunos subgrupos H tiene puntos fijos obtenemos:

Teorema. *Si H es un grupo de operadores inversibles en $\mathcal{B}_2(\mathcal{H})+\mathbb{C}1$ tal que $sup_{h\in H}\|hh^*-1\|_2 < \infty$ entonces existe un s en P tal que $s^{-1}Hs$ es un grupo de operadores unitarios.*

Algunos de los resultados presentados en esta tesis fueron publicados en revistas internacionales como artículos de los cuales soy el único autor [42, 43].

Precedentes

Los resultados en esta tesis tienen precendentes en los siguientes trabajos:

- Los teoremas de descomposición tienen como precedente la decomposición polar de operadores. En 1955 Mostow [46] munió al conjunto de matrices positivas inversibles con una metrica Riemanniana de curvatura negativa. Usando esta métrica Mostow construyó entornos tubulares globales de subvariedades totalmente geodésicas, donde la noción de vector normal a la subvariedad está dada por el producto interno de Hilbert-Schmidt. Este resultado fue extendido por Larotonda en [37] al contexto de perturbaciones Hilbert-Schmidt de la identidad. Corach, Porta y Recht estudiaron la geometría de curvatura no positiva del cono de operadores positivos e inversibles de un álgebra C^* en [20, 21, 22, 23]. Basados en estos trabajos Porta y Recht demostraron un teorema de descomposición en [57]; en este trabajo la variedad y la subvariedad son los operadores positivos e inversibles de un álgebra A y una subálgebra B respectivamente, y la noción de vector normal a la subvariedad es dada por el núcleo de una esperanza condicional $E : A \to B$. En [15] Conde y Larotonda extendieron este teorema al contexto de espacios simétricos G/U modelizados en espacios de Banach.

- En 1955 [45] Mostow usó el teorema de descomposición obtenido en [46] para probar que un espacio homogéneo con grupo asociado G cuyo subgrupo de isotropía es conexo y autoadjunto (módulo el radical de G) admite un fibrado covariante, i.e. es isomorfo a un fibrado vectorial asociado. En [10] este fibrado covariante fue usado por Bielawski para construir un isomorfismo entre el fibrado tangente de G/K y la complexificación de G/K, donde G/K es un espacio localmente simétrico de tipo compacto con K conexo. Un fibrado análogo fue construido por Beltiţă y Galé en [6] en el contexto de álgebras C^* usando el teorema de descomposición de Porta y Recht. Aquí los espacios homogéneos son variedades Grassmannianas generalizadas U_A/U_B, donde U_A y U_B son los grupos unitarios de álgebras C^* relacionadas por una esperanza condicional $E : A \to B$. Como consecuencia se obtiene un isomorfismo $T(U_A/U_B) \simeq G_A/G_B$, donde G_A y G_B son los grupos de operadores inversibles del álgebra A y de la subálgebra B respectivamente.

- El estudio geométrico de espacios de representaciones es un área de investigación activa, véase [31] para el caso de dimensión finita. Aquí los espacios de representaciones son munidos con la estructura de variedad topológica o algebráica y los problemas principales son la determinación de las componentes conexas y las clausuras de órbitas. En el contexto de dimensión infinita Andruchow, Corach y Stojanoff demostraron que álgebras de operadores son inyectivas o nucleares si los correspondientes espacios de representaciones son espacios homogéneos reductivos, véase [41]. Esta línea de investigación fue continuada por Corach y Galé en [18, 19] donde diagonales virtuales de álgebras de Banach proveen formas de conexión en los espacios de representaciones, véase el artículo [30] de Galé y el Capítulo 8 del libro de Runde [58] para mayor información.

- La pregunta sobre cuáles grupos uniformemente acotados de $\mathcal{B}(\mathcal{H})$ son similares a grupos de unitarios tiene una larga historia. Un resultado antiguo de teoría de representaciones afirma que si $H \subseteq \mathcal{B}(\mathbb{C}^n)$ es un grupo uniformemente acotado entonces es similar a un grupo de matrices unitarias. Dado que la clausura del grupo es compacta, esta tiene una medida de Haar bi-invariante y el unitarizante se obtiene como la raiz cuadrada del promedio de $\{hh^*\}_{h\in H}$. Posteriormente Elie Cartan demostró que grupos de Lie semisimples G admiten (módulo conjugación) un único subgrupo compacto maximal K, usando que G/K es una variedad Riemanniana de curvatura negativa y el teorema de punto fijo de Cartan, véase [34, I. 13 y VI. 2]. Szokefalvi-Nagy [60, Teorema I] demostró que toda representación uniformemente acotada $\mathbb{Z} \to \mathcal{B}(\mathcal{H})$ es unitarizable. Este resultado fue extendido por Day [25], Dixmier [26], Nakamura y Takeda [47] a toda representación uniformemente acotada de un grupo topológico promediable, promediando sobre la media invariante. Otros enfoques en el contexto de dimensión infinita no involucran representaciones, véanse los artículos de Ostrovkii, Shulman, Turowska, Vasilescu y Zsido [50, 64].

Introduction

Infinite-dimensional homogeneous spaces and non-positive curvature

In recent years, the geometrical study of operator algebras and their homogeneous spaces has become a central topic in the study of infinite dimensional geometry. It is a source of examples and counterexamples, and the operator algebra techniques (Banach algebras and C^* algebras, with their distinguished tools) are being used for obtaining results on abstracts infinite dimensional manifolds by studying their groups of automorphism, isometries, and their associated fiber bundles and G-bundles. The reader can see the recent book [9] by D. Beltiţă for a full account of these objects and a comprehensive list of references, see also the section "Precedents" at the end of this section.

A homogeneous space for a group G is a manifold on which the group G acts transitively, i.e. an orbit. It can be alternatively be viewed as a quotient G/H of a Banach-Lie group G by a Lie subgroup H. In the case where the homogeneous space is the manifold of positive invertible operators of an operator algebra (endowed with a Finsler structure making it negatively curved) decomposition theorems extending the usual polar decomposition can be proved. With this decomposition theorems we endow the complexifications of certain homogeneous spaces with the structure of associated vector bundles, and with these associated vector bundle structures or fiberings, we define adapted complex structures on tangent bundles of coadjoint orbits in operator ideals, and unitary similarity orbits of system of projections (Flag manifolds) and partial isometries (Stiefel manifolds).

Using properties of the manifold of positive invertible operators such as the convexity of the distance along geodesics, the minimanility of projections onto submanifolds and the existence of circumcenters of bounded sets, we study similarity problems from a geometrical perspective. Similarity problems ask in different contexts when a group H of invertible bounded operators acting on a Hilbert space is conjugate to a group of unitaries. Other related questions are about the positive invertible operator s such that $s^{-1}Hs$ is a group of unitary operators. If a group of bounded invertible operators on a Hilbert space is conjugate to a group of unitaries then it is uniformly bounded. The converse does not hold in general, so further assumptions on the group have to be made. One variant of this problem is to look at a unital algebra homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$, where A is a C^*-algebra, and study under what conditions on the map π and the algebra A, the image of the unitary group under π is unitarizable. In this case the orbits of representations are homogeneous spaces for the natural conjugation action $g \cdot \pi = g\pi(\cdot)g^{-1}$, where π is a representation and g is an invertible operator. See [54] for further information about similarity problems.

Main results

A few words about notation are in order. We use M, N to denote manifolds and the letters x, y, z to denote its points. For a smooth map between manifolds $f : M \to N$ we use the notation $f_* : TM \to TN$ for the tangent map and $f_{*x} : T_xM \to T_{f(x)}N$ for the tangent map at $x \in M$. If $\alpha : I \to M$ is a smooth curve then we define as usual $\dot{\alpha}(t) = \alpha_{*t}(\frac{d}{dt})$. We denote vectors fields with greek letters ξ, λ and homomorphisms with greek letters π, ρ. The capital letters X, Y, Z will be reserved for vectors. German characters $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{p}$ will be used to denote Lie algebras and subspaces of Lie algebras. We denote with G, H, U groups and with g, h, u, v its elements. The first letters of the alphabet a, b, c will be reserved for positive invertible operators. We denote by V, W and Z Banach spaces and with U open subsets of these Banach spaces when we consider them as local charts.

In Chapter 1 we introduce basic results of Lie theory and results about infinite dimensional negatively curved symmetric spaces which will help understand the other chapters. A Finsler symmetric space of semi-negative curvature $M = G/U$ is defined as a quotient G/U, where G is a Banach-Lie group, U is the fixed point set of an involution $\sigma : G \to G$ and $\| \cdot \|$ is an Ad_U-invariant norm on $\mathfrak{p} = Ker(\sigma_{*1} + 1) \simeq T_{1U}(G/U)$ which gives G/U a Finsler structure such that the diferential of the exponential map at every point is an expansive operator. We denote $M = G/U = Sym(G, \sigma, \| \cdot \|)$.

In Chapter 2 we address decompositions of Banach symmetric spaces and complexifications of homogeneous spaces modeled on Banach-Lie groups. In Section 2.2 we recall the category of reductive pairs introduced by Belţiţă and Galé [7]. A reductive structure with involution is a quadruple $(G_A, G_B; E, \sigma)$ such that:

- G_B is a Lie subgroup of the Banach-Lie group G_A
- $E : \mathfrak{g}_A \to \mathfrak{g}_B$ is a linear map between the Lie algebras of the Lie groups G_A and G_B such that $Ad_g \circ E = E \circ Ad_g$ for every $g \in G_B$.

- $\sigma : G_A \to G_A$ is an involution such that $\sigma(G_B) = G_B$ and $\sigma_{*1} \circ E = E \circ \sigma_{*1}$.

Using this category and a global tubular neighborhood theorem proved by Conde and Larotonda [16] a polar decomposition for nested finite sequences of reductive pairs of Banach-Lie groups is obtained:

Theorem. *If for $n \geq 2$ we have the following inclusions of connected Banach-Lie groups, the following maps between their Lie algebras*

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n$$

$$\mathfrak{g}_1 \xleftarrow{E_2} \mathfrak{g}_2 \xleftarrow{E_3} \ldots \xleftarrow{E_n} \mathfrak{g}_n$$

and a morphism $\sigma : G_n \to G_n$ such that:

- $(G_n, G_{n-1}; E_n, \sigma)$,$(G_{n-1}, G_{n-2}; E_n, \sigma|_{G_{n-1}})$,..., $(G_2, G_1; E_2, \sigma|_{G_2})$ *are reductive structures with involution.*
- $M_n = G_n/U_n = Sym(G_n, \sigma, \|\cdot\|)$ *is a simply connected Finsler symmetric space of semi-negative curvature.*
- $\|E_k|_{\mathfrak{p}_k}\| = 1$ *for $k = 2, \ldots, n$, where we use the norm of the previous item restricted to $\mathfrak{p}_k := \mathfrak{p} \cap \mathfrak{g}_k$.*

Then the maps

$$\Phi_n : U_n \times \mathfrak{p}_{E_n} \times \cdots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1 \to G_n$$
$$(u_n, X_n, \ldots, X_2, Y_1) \mapsto u_n e^{X_n} \ldots e^{X_2} e^{Y_1}$$

$$\Psi_n : \mathfrak{p}_{E_n} \times \cdots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1 \to G_n^+$$
$$(X_n, \ldots, X_2, Y_1) \mapsto e^{Y_1} e^{X_2} \ldots e^{X_{n-1}} e^{2X_n} e^{X_{n-1}} \ldots e^{X_2} e^{Y_1}$$

are diffeomorphisms, where $\mathfrak{p}_{E_k} := KerE_k \cap \mathfrak{p}_k$ for $k = 2, \ldots, n$.

In Section 2.3 the complexifications of some homogeneous spaces are studied. If G_B is a subgroup of the Banach-Lie group G_A and σ is an involution on G_A leaving G_B invariant, then under certain conditions the quotient U_A/U_B of the respective fixed point subgroups of G_A and G_B is a submanifold $U_A/U_B \hookrightarrow G_A/G_B$ which is the fixed point set of the involution $\sigma_G : G_A/G_B \to G_A/G_B$, $gG_B \mapsto \sigma(g)G_B$. Therefore the complex manifold G_A/G_B can be considered as a complexification of the real manifold U_A/U_B. The decomposition theorem is used to give the complexifications G_A/G_B of a homogeneous space U_A/U_B the structure of an associated vector bundle:

Theorem. *Let $M_A = G_A/U_A = Sym(G_A, \sigma, \|\cdot\|)$ be a simply connected Finsler symmetric space of semi-negative curvature and $(G_A, G_B; E, \sigma)$ a reductive structure with involution such that $\|E|_{\mathfrak{p}}\| = 1$. Consider $\Psi_0^E : U_A \times \mathfrak{p}_E \to G_A$, $(u, X) \mapsto ue^X$ and $\kappa : (u, X) \mapsto [(u, X)]$ the quotient map. Then there is a unique real analytic, U_A-equivariant diffeomorphism $\Psi^E : U_A \times_{U_B} \mathfrak{p}_E \to G_A/G_B$ such that the diagram*

$$\begin{array}{ccc} U_A \times \mathfrak{p}_E & \xrightarrow{\Psi_0^E} & G_A \\ \kappa \downarrow & & \downarrow q \\ U_A \times_{U_B} \mathfrak{p}_E & \xrightarrow{\Psi^E} & G_A/G_B \end{array}$$

commutes, where $q : G_A \to G_A/G_B$, $g \mapsto gG_B$ is the canonical quotient map.

Therefore the homogeneous space G_A/G_B has the structure of an U_A-equivariant fiber bundle over U_A/U_B with the projection given by the composition

$$G_A/G_B \xrightarrow{(\Psi^E)^{-1}} U_A \times_{U_B} \mathfrak{p}_E \xrightarrow{\ \Xi\ } U_A/U_B$$

$$ue^X G_B \mapsto [(u, X)] \mapsto uU_B \qquad \textit{for } u \in U_A \textit{ and } X \in \mathfrak{p}_E$$

and typical fiber $\mathfrak{p}_E$.

This theorem is used to construct under certain assumptions an isomorphism $G_A/G_B \simeq T(U_A/U_B)$ between the complexification and tangent space of homogeneous spaces U_A/U_B:

Corollary. *Assume the conditions of the previous theorem are satisfied and assume that G_A is a complex Banach-Lie group, E is $\mathbb{C}$-linear and $\mathfrak{u} = i\mathfrak{p}$. Then*

$$G_A/G_B \xrightarrow{(\Psi^E)^{-1}} U_A \times_{U_B} \mathfrak{p}_E \xrightarrow{\Theta} U_A \times_{U_B} \mathfrak{u}_E \xrightarrow{\alpha^E} T(U_A/U_B)$$

$$ue^X G_B \mapsto [(u, X)] \mapsto [(u, iX)] \mapsto (\mu_u)_{*o} q_{*1}(iX)$$

is a U_A-equivariant diffeomorphism between the complexification G_A/G_B and the tangent bundle $T(U_A/U_B)$ of the homogeneous space U_A/U_B. Here $\mu_u : U_A/U_B \to U_A/U_B$, $vU_B \mapsto uvU_B$ is a translation, $\mathfrak{u}$ is the Lie algebra of U_A and $\mathfrak{u}_E = KerE \cap \mathfrak{u}$. Under the above identification the involution $\sigma_G : G_A/G_B \to G_A/G_B$, $gG_B \mapsto \sigma(g)G_B$ is the map $T(U_A/U_B) \to T(U_A/U_B)$, $V \mapsto -V$.

Therefore for a class of smooth homogeneous spaces of Banach-Lie groups their tangent bundles can be endowed with a complex manifold structure. In this case, the map between their tangent bundles given by $V \mapsto -V$ is anti-holomorphic as in the adapted complex structures studied by Lempert and his co-workers, see [39]. Examples of these homogeneous spaces are coadjoint orbits in p-Schatten ideals, flag manifolds, and Stiefel manifolds in the context of operator algebras, see [8, 14, 27].

In Chapter 3 a new approach of geometrical nature to similarity problems is developed. The main contribution here is related to the analysis in different contexts of the orbit structure of the natural isometric action of subgroups H of the group of invertible elements on the cone P of positive invertible operators of an operator algebra. This action is given by $h \cdot a = hah^*$ with $h \in H$ and $a \in P$.

In Section 3.3 the convexity of the distance along geodesics in the cone of positive invertible operators is used to prove the following geometric inequality:

Proposition. *If $\pi : A \to \mathcal{B}(\mathcal{H})$ is a completely bounded unital homomorphism between a C^*-algebra A and the algebra of bounded operators on a Hilbert space $\mathcal{H}$, and s is a positive invertible operator that minimizes $\|s\|\|s^{-1}\|$ among the positive invertible operators such that $Ad_s \circ \pi = s\pi(\cdot)s^{-1}$ is a $*$-representation, then*

$$\|Ad_{s^t} \circ \pi\| \leq \|\pi\|^{1-t} \text{ and } \|Ad_{s^t} \circ \pi\|_{c.b.} = \|\pi\|_{c.b.}^{1-t},$$

where $\| \cdot \|_{c.b.}$ is the completely bounded norm of a homomorphism.

This result was proved by Pisier in [55] using complex interpolation techniques. Also, minimality properties of projections to closed convex sets in the cone P are used to prove minimality properties of canonical unitarizers of unital homomorphisms $\pi = g\rho(\cdot)g^{-1}$. Here g is an invertible operator in $\mathcal{B}(\mathcal{H})$ and $\rho : A \to \mathcal{B}(\mathcal{H})$ is a $*$-representation of a C^*-algebra A such that there is a conditional expectation $E : \mathcal{B}(\mathcal{H}) \to \rho(A)'$. These canonical unitarizers were obtained by Andruchow, Corach and Stojanoff in [2].

Theorem. *If $\|I - E\| = 1$ then the canonical positive invertible s making the unital homomorphism $Ad_s \circ \pi : A \to \mathcal{B}(\mathcal{H})$ a $*$-representation satisfies $\|s\|\|s^{-1}\| = \|\pi\|_{c.b.}$, i.e. it minimizes the quantity $\|r\|\|r^{-1}\|$ among the positive invertible r such that $Ad_r \circ \pi$ is a $*$-representation.*

In Section 3.4 we address the question of existence of unitarizers of groups of invertible operators H, i.e. positive invertibles s such that sHs^{-1} is a group of unitaries, when these groups act on manifolds P of positive invertible operators endowed with a metric derived from a trace. Here the Bruhat-Tits fixed point theorem is used to show that the square root of the circumcenter of $\{hh^*\}_{h\in H}$ in P is a unitarizer of H. In the case of a finite von Neumann algebra we obtain the following existence result proved previously in [64] using different techniques:

Theorem. *If H is a group of invertible operators in a finite von Neumann algebra A such that $\sup_{h\in H} \|h\| = |H| < \infty$ then there is an $s \in \{a \in A : |H|^{-1}1 \leq a \leq |H|1\}$ such that $s^{-1}Hs$ is a group of unitary operators in A.*

In this case the submanifolds normal to the set of fixed point are shown to be invariant under the action $h \cdot a = hah^*$. If $\mathcal{B}_2(\mathcal{H})$ is the ideal of Hilbert-Schmidt operators then by proving that the canonical action of $G = \{g \in \mathcal{B}_2(\mathcal{H}) + \mathbb{C}1 : g \text{ is invertible}\}$ on $P = \{g \in \mathcal{B}_2(\mathcal{H}) + \mathbb{C}1 : g > 0\}$ restricted to some subgroups H has a fixed point we obtain the following result:

Theorem. *If H is a group of invertible operators in $\mathcal{B}_2(\mathcal{H}) + \mathbb{C}1$ such that $sup_{h \in H} \|hh^* - 1\|_2 < \infty$ then there is an s in P such that $s^{-1}Hs$ is a group of unitaries.*

Some of the results in this thesis have been published in research articles [42, 43], for which I am the sole author.

Precedents

The results in this thesis have precedents in the following works:

- Decomposition theorems have precedents in the polar decomposition of operators. In 1955 Mostow [46] endowed the set of positive invertible matrices with a Riemannian metric of negative curvature. Using this metric Mostow constructed global tubular neighborhoods to totally geodesic submanifolds where the notion of normal vector to the submanifold is provided by the Hilbert-Schmidt inner product. This result was extended by Larotonda in [37] to the context of Hilbert-Schmidt perturbation of the identity operator. Corach, Porta and Recht studied the non-positively curved geometry of the cone of positive invertible elements in C^*-algebras in [20, 21, 22, 23]. Based on this work Porta and Recht proved a decomposition theorem in [57]; here the manifold and submanifold are the positive invertible elements of an algebra A and a subalgebra B respectively, and the notion of normal vector to the submanifold is provided by the kernel of a conditional expectation $E : A \to B$. In [15] Conde and Larotonda extended this theorem to the context of symmetric spaces G/U modeled on Banach spaces.

- In 1955 [45] Mostow used the decomposition theorem obtained in [46] to prove that a homogeneous space with associated group G whose isotropy subgroup is connected and selfadjoint (modulo the radical of G) admits a covariant fibering, i.e. is isomorphic to an associated vector bundle. In [10] this covariant fibering was used by Bielawski to construct an isomorphism between the tangent bundle of G/K and a complexification of G/K, where G/K is a locally symmetric space of compact type with K connected. An analogous fibering was constructed by Beltiţă and Galé in [6] in the context of C^*-algebras using the decomposition theorem of Porta and

Recht. Here the homogeneous spaces are generalized Grassmann manifolds U_A/U_B, where U_A and U_B are the unitary groups of C^*-algebras related by a conditional expectation $E : A \to B$. Hence, an isomorphism $T(U_A/U_B) \simeq G_A/G_B$ is obtained, where G_A and G_B are the groups of invertible elements of the algebra A and the subalgebra B respectively.

- The geometrical study of spaces of representations is an active area of research, see [31] for the finite dimensional case. Here the spaces of representations are endowed with the structure of a topological or algebraic manifold and the main problems are to determine the connected components and the closure of orbits. In the infinite dimensional setting Andruchow, Corach and Stojanoff proved that operator algebras are injective or nuclear if the corresponding space of representations are homogeneous reductive spaces, see [41]. This line of research was continued by Corach and Galé in [18, 19] where virtual diagonals of Banach algebras provide connection forms in the spaces of representations, see [30] by Galé and [58, Chapter 8] by Runde for further information.

- The question of which uniformly bounded subgroups of $\mathcal{B}(\mathcal{H})$ are similar to groups of unitaries has a long history. An old result of representation theory states that if $H \subseteq \mathcal{B}(\mathbb{C}^n)$ is a uniformly bounded subgroup, then it is similar to a group of unitaries. Since the closure of the group is compact it has a bi-invariant Haar measure and the unitarizer is obtained as the square root of the average of $\{hh^*\}_{h\in H}$. Later Elie Cartan showed that a semisimple Lie group G admits up to conjugacy a unique maximal compact subgroup K using the fact that G/K is a Riemannian manifold of negative curvature and the Cartan fixed point theorem, see [34, I. 13 and VI. 2]. Szokefalvi-Nagy [60, Theorem I] showed that any uniformly bounded representation $\mathbb{Z} \to \mathcal{B}(\mathcal{H})$ is unitarizable. This was extended by Day [25], Dixmier [26], Nakamura and Takeda [47] to any uniformly bounded representation of an amenable topological group, via averaging over an invariant mean. Other approaches in the infinite-dimensional context do not involve representations, see [50, 64] by Ostrovskii, Shulman, Turowska, Vasilescu and Zsido.

Contents

Chapter 1

Preliminaries

1.1 Introduction

In Sections 1.2 and 1.3 the reader can find basic information about Lie theory, fibre bundles and connections. General references for differential geometry in the context of Banach manifolds are [1, 36].

In Section 1.4 symmetric spaces are introduced and the basic properies of its cannonical connection and exponential are presented. The basic example of symmetric space, the quotient G/U where U is the fixed point space of an involution σ on a Banach-Lie group G, is analyzed. Special features when $G/U = P$ is the space of positive invertible elements in a C^*-algebra and G and U are the groups of invertible and unitary elements respectively are shown. The connection derived in this section is the same one as the one derived in [22] as the horizontal invariant subspaces of a principal bundle $G \to G/U = P$.

In Section 1.5 norms on tangent spaces of manifolds which make parallel transport isometric are introduced. These norms determine distance functions and we analize different compatibility conditions between the topology of the manifolds, norms and distance. We derive equations for the tangent norms in the case $G/U = P$. The canonical action of G on $G/U = P$ is shown to be isometric, a fact that implies that parallel transport is isometric.

In Section 1.6 the property of semi-negative curvature for manifolds with certain connections and compatible tangent norms is defined. We present some consequences of this property such as the exponential metric increasing property $\|X-Y\| \leq d(exp(X), exp(Y))$ and the the convexity of the distance along two geodesics.

1.2 Lie theory

1.2.1 Lie groups and Lie algebras

We denote by $\mathcal{V}(G)$ the set of vector fields on G. If G is a Lie group we say that a vector field $\xi \in \mathcal{V}(G)$ is left invariant whenever for all $h \in G$ the diagram

$$\begin{array}{ccc} TG & \xrightarrow{(L_h)_*} & TG \\ {\scriptstyle\xi}\uparrow & & \uparrow{\scriptstyle\xi} \\ G & \xrightarrow{L_h} & G \end{array}$$

commutes, that is

$$\xi_{L_h g} = (L_h)_{*g}\xi_g$$

for all $g, h \in G$. We denote by $\mathcal{V}_l(G)$ the set of all left invariant vector fields on G. The map

$$\iota : T_1G \to \mathcal{V}_l(G), \quad \iota(X)_g = (L_g)_{*1}X \in T_gG$$

for $X \in T_1G$ and $g \in G$ is a linear isomorphism with inverse

$$\iota^{-1} : \mathcal{V}_l(G) \to T_1(G), \quad \xi \mapsto \xi_1.$$

If $\xi, \zeta \in \mathcal{V}_l(G)$ then $[\xi, \zeta] \in \mathcal{V}_l(G)$, where

$$\mathcal{V}(G) \times \mathcal{V}(G) \to \mathcal{V}(G), \quad (\xi, \zeta) \to [\xi, \zeta]$$

is the Lie bracket of vector fields. By means of ι we can define a bracket in T_1G such that

$$\iota([X, Y]) = [\iota(X), \iota(Y)]$$

for $X, Y \in T_1G$. This bracket is bilinear, antisymmetric and satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in T_1G$. The tangent space T_1G at the identity of a group G with the bracket operation is the *Lie algebra* of the group and is denoted by $\mathfrak{g}$.

Definition 1.2.1. *A Lie group homorphism $\phi : \mathbb{R} \to G$ is called a* ***1-parameter subgroup*** *of G. Given $X \in \mathfrak{g}$ there is a unique 1-parameter subgroup*

$$exp_X : \mathbb{R} \to G$$

such that $exp_X(0) = 1$ *and* $(\dot{exp_X)}(0) = X$. *We define the* ***exponential map***

$$exp : \mathfrak{g} \to G$$

by setting

$$exp(X) = exp_X(1).$$

Definition 1.2.2. *A map* $\alpha : \mathfrak{g} \to \mathfrak{h}$ *between two Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$ *is a* ***Lie algebra homomorphism*** *if*

$$\alpha([X, Y]) = [\alpha(X), \alpha(Y)]$$

for all $X, Y \in \mathfrak{g}$.

Proposition 1.2.3. *If* $\phi : G \to H$ *is a Lie group homomorphism, then* $\phi_{*1} : \mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra homomorphism.*

Example 1.2.4. *Let* Z *be a real Banach space and* $A = \mathcal{B}(Z)$ *the unital associative real Banach algebra of all bounded linear operators on* Z. *Hence the group*

$$GL(Z) = \{g \in \mathcal{B}(Z) : g \text{ is invertible } \}$$

is a Banach-Lie group whose Lie algebra is $\mathcal{B}(Z)$ *with bracket defined by* $[X, Y] = XY - YX$ *whenever* $X, Y \in \mathcal{B}(Z) \simeq T_1(GL(Z))$. *In this case the exponential is the usual exponential given by power series*

$$exp : \mathcal{B}(Z) \to GL(Z), \quad exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Theorem 1.2.5. *Let* $\phi : G \to H$ *be a Lie group homomorphism, then the following diagram is commutative:*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \uparrow{\scriptstyle exp} & & \uparrow{\scriptstyle exp} \\ \mathfrak{g} & \xrightarrow{\phi_{*1}} & \mathfrak{h} \end{array}$$

Proof. If $X \in \mathfrak{g}$ then $t \mapsto \phi(exp(tX))$ is a one-parameter subgroup of H whose tangent at 0 is $\phi_{*1}(X)$. But $t \mapsto exp(t\phi_{*1}(X))$ is the unique 1-parameter subgroup of H whose tangent at 0 is $\phi_{*1}(X)$. Thus

$$\phi(exp(tX)) = exp(t\phi_{*1}(X))$$

for all $t \in \mathbb{R}$. Hence at $t = 1$

$$\phi(exp(X)) = exp(\phi_{*1}(X)).$$

□

1.2.2 Group actions

Definition 1.2.6. *Let X be a set and let G be a group. A map $\mu : G \times X \to X$ such that*

$$\mu(gh, x) = \mu(g, \mu(h, x)), \quad \mu(1, x) = x$$

*for all $g, h \in G$ and $x \in X$ is called an **action** of G on X and we call X a G-set. We usually write $\mu(g, x) = g \cdot x$. For a fixed $g \in G$ the map $x \mapsto \mu(g, x)$ is a biyection of X wich we shall denote by μ_g. For $x \in X$ we define the **stabilizer** of x as the group*

$$Stab(x) = \{g \in G : g \cdot x = x\}$$

*and the **orbit** of x as the set*

$$\mathcal{O}_G(x) = \{g \cdot x : g \in G\}.$$

There is a bijection $G/Stab(x) \simeq \mathcal{O}_G(x)$ given by

$$G/Stab(x) \to \mathcal{O}_G(x), \quad gStab(x) \mapsto g \cdot x.$$

For $g \in G$ we have

$$Stab(g \cdot x) = gStab(x)g^{-1}.$$

*An action is called **transitive** if for $x, y \in X$ there is a $g \in G$ such that $g \cdot y = x$. This is equivalent to $\mathcal{O}_G(x) = X$ for all $x \in X$. An action is called **free** if $g \cdot x = h \cdot x$ for $g, h \in G$ and $x \in X$ then $g = h$. A map $\Psi : X \to Y$ between two G-spaces is called G-**equivariant** if for all $g \in G$ and $x \in X$*

$$\Psi(g \cdot x) = g \cdot \Psi(x).$$

If M is a manifold and G is a Lie group, an action $\mu : G \times M \to M$ which is smooth, i.e. C^∞, is called a ***smooth action*** of G on M. If M is a linear space and each μ_g is bounded linear, then $G \to GL(M)$, $g \mapsto \mu_g$ is a ***representation*** of G.

Lemma 1.2.7. *Let μ be a smooth action of G on M and assume that x is a fixed point of the action. The the map*

$$\psi : G \to GL(T_xM)$$

defined by

$$\psi_g = (\mu_g)_{*x} : T_xM \to T_xM$$

is a representation of G.

For $g \in G$ let L_g and R_g stand for the *left and right translation diffeomorphisms* on G defined by $L_g h = gh$ and $R_g h = hg$ for $h \in G$. A Lie group G acts on itself by ***inner automorphisms***:

$$I : G \times G \to G, \quad I(g,h) = I_g(h) = ghg^{-1} = L_g R_{g^{-1}} h = R_{g^{-1}} L_g h.$$

The identity is a fixed point of this action, hence the map

$$G \to GL(\mathfrak{g}), \quad g \mapsto (I_g)_{*1}$$

is a representation of G. This is called the ***adjoint representation*** and is denoted by

$$Ad : G \to GL(\mathfrak{g}).$$

We let the differential of the adjoint representation at the identity be denoted by ad,

$$ad = Ad_{*1} : T_1 G = \mathfrak{g} \to \mathcal{B}(\mathfrak{g}).$$

With the canonical bracket in $\mathcal{B}(\mathfrak{g})$ described in Example 1.2.4 ad is a morphism of Lie algebras, i.e.

$$ad_{[X,Y]} = [ad_X, ad_Y] = ad_X ad_Y - ad_Y ad_X$$

for $X, Y \in \mathfrak{g}$. We denote $Ad(g)$ by Ad_g and $ad(X)$ by ad_X.

Proposition 1.2.8. *If G is a Lie group*

$$ad_X Y = [X, Y]$$

for $X, Y \in \mathfrak{g}$ $ad_X Y = [X, Y]$.

Applying Theorem 1.2.5 to the automorphism I_g of G we get

Proposition 1.2.9. *If G is a Lie group then for $g \in G$*

$$\begin{array}{ccc} G & \xrightarrow{I_g} & G \\ \uparrow{\scriptstyle exp} & & \uparrow{\scriptstyle exp} \\ \mathfrak{g} & \xrightarrow{Ad_g} & \mathfrak{g} \end{array}$$

commutes, or

$$exp(Ad_g(X)) = I_g(exp(X)) = g(exp(X))g^{-1}$$

for $g \in G$ and $X \in \mathfrak{g}$.

Also applying Theorem 1.2.5 to the homomorphism of Lie groups $Ad : G \to GL(\mathfrak{g})$ we get

Proposition 1.2.10. *If G is a Lie group*

$$\begin{array}{ccc} G & \xrightarrow{Ad} & GL(\mathfrak{g}) \\ {\scriptstyle exp}\uparrow & & \uparrow{\scriptstyle exp} \\ \mathfrak{g} & \xrightarrow{ad} & \mathcal{B}(\mathfrak{g}) \end{array}$$

commutes, or

$$e^{ad_X} = Ad_{exp(X)}$$

for $X \in \mathfrak{g}$.

Example 1.2.11. *In the case of $GL(Z)$ for a Banach space Z the adjoint representation is given by*

$$Ad : GL(Z) \to \mathcal{B}(\mathcal{B}(Z)), \quad Ad_g X = gXg^{-1}$$

and

$$e^{Ad_g X} = e^{gXg^{-1}} = ge^X g^{-1} = I_g(e^X)$$

for $g \in GL(Z)$ and $X \in \mathcal{B}(Z) = T_1 GL(Z)$.

1.2.3 Principal and associated fibre bundles

If U is a Lie subgroup of a Lie group G in the sense that it is a submanifold of the manifold G, then the Lie algebra $\mathfrak{u}$ of U has a complement $\mathfrak{p}$ in $\mathfrak{g}$, i.e. $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$. Therefore the quotient space $M = G/U$ has a Banach manifold structure and the quotient map

$$q : G \to G/U = M, \quad g \to q(g) = gU$$

is a submersion. For $h \in G$, let

$$\mu_h : M \to M, \quad \mu_h(q(g)) = q(hg) = q(L_h g)$$

for $g \in G$. Differentiating the last equation in $g \in G$ we get

$$(\mu_h)_{*q(g)} q_{*g} = q_{*hg}(L_h)_{*g}.$$

The action of G on M given by $h \cdot q(g) = \mu_h(q(g))$ is smooth and transitive.

The maps $q_{*1} : \mathfrak{p} \to T_o M$ and $(\mu_g)_{*o} : T_o M \to T_{q(g)} M$ for $g \in G$ are isomorphisms so that a generic vector in $T_{q(g)} M$ will be denoted by $(\mu_g)_{*o} q_{*1} X$ with $X \in \mathfrak{p}$.

Definition 1.2.12. *A* ***principal G-bundle****, where G denotes a Lie group, is a fiber bundle $\pi : P \to X$ together with a continuous right action of G on P which preserves the fibers and acts freely and transitively on them. This implies that each fiber of the bundle is diffeomorphic to the group G itself.*

Note that $q : G \to G/U = M$ is a principal U-bundle. If $\mathfrak{p}$ is Ad_U-invariant, then restricting the adjoint representation there is a representation $Ad : U \to \mathcal{B}(\mathfrak{p})$ and U acts on $G \times \mathfrak{p}$ by $u \cdot (g, X) = (gu^{-1}, Ad_u X)$ for $u \in U$ and $(g, X) \in G \times \mathfrak{p}$. We denote by $[g, X]$ the orbit of (g, X) and by $G \times_U \mathfrak{p}$ the orbit space which is a smooth manifold. In this case there is an ***associated vector bundle***

$$\pi : G \times_U \mathfrak{p} \to G/U, \quad [g, X] \to gU = q(g)$$

with typical fiber $\mathfrak{p}$. Note that G acts on $G \times_U \mathfrak{p}$ by $g \cdot [h, X] = [gh, X]$ and on G/U by $g \cdot hU = ghU$. With these actions the quotient map π is G-equivariant.

Theorem 1.2.13. *If $\mathfrak{p}$ is Ad_U-invariant then there is a G-equivariant vector bundle isomorphism from the associated vector bundle $G \times_U \mathfrak{p} \to G/U$ onto the tangent bundle $T(G/U) \to G/U$ given by*

$$\Delta : G \times_U \mathfrak{p} \to T(G/U), \qquad [(u, X)] \mapsto (\mu_u)_{*o} q_{*1} X,$$

where the action of G on $T(G/U)$ is given by $u \cdot - = (\mu_u)_* -$ for every $u \in G$.

Proof. Let $\delta : G \times G/U \to G/U$ be given by $(g, hU) \mapsto ghU$, then $\partial_2 \delta : G \times T(G/U) \to T(G/U)$, $(g, V) \mapsto (\mu_g)_* V$. Since $\mathfrak{p} \simeq T_o(G/U)$, $X \mapsto q_{*1} X$ restricting $\partial_2 \delta$ to $G \times T_o(G/U)$ we get a map $\Delta_0 : G \times \mathfrak{p} \to T(G/U)$, $(g, X) \mapsto (\mu_g)_{*o} q_{*1} X$.

We assert that there is a unique G-equivariant diffeomorphism $\Delta : G \times_U \mathfrak{p} \to T(G/U)$ such that $\Delta \circ \kappa = \Delta_0$, where κ is the quotient map $(g, X) \mapsto [(g, X)]$.

To prove that Δ is well defined we see that for every $g \in G$, $u \in U$ and $X \in \mathfrak{p}$

$$\begin{aligned} \Delta_0(u \cdot (g, X)) &= \Delta_0(gu^{-1}, Ad_u X) = (\mu_{gu^{-1}})_{*o} q_{*1} Ad_u X \\ &= (\mu_{gu^{-1}})_{*o} q_{*1} (I_u)_{*1} X = (\mu_{gu^{-1}} q I_u)_{*1} X \\ &= (\mu_g \mu_{u^{-1}} q L_u R_{u^{-1}})_{*1} X = (\mu_g q L_{u^{-1}} L_u R_{u^{-1}})_{*1} X \\ &= (\mu_g q R_{u^{-1}})_{*1} X = (\mu_g q)_{*1} = (\mu_g)_{*o} q_{*1} X = \Delta_0(g, X) \end{aligned}$$

The uniqueness of Δ is a consequence of the surjectivity of κ. Note that Δ is surjective because $(\mu_g)_{*o} : T_o(G/U) \to T_{q(g)}(G/U)$ is bijective for every $g \in G$. To see that Δ is injective assume that $(\mu_{g_1})_{*o}q_{*1}X_1 = (\mu_{g_2})_{*o}q_{*1}X_2$. Then $q(g_1) = q(g_2)$ and therefore there is a $u \in U$ such that $g_1 u = g_2$. Then

$$\begin{aligned}(\mu_{g_1})_{*o}q_{*1}X_1 &= (\mu_{g_2})_{*o}q_{*1}X_2 = (\mu_{g_1 u}q)_{*1}X_2 = (\mu_{u_1}\mu_u q)_{*1}X_2 \\ &= (\mu_{g_1}\mu_u q R_{u^{-1}})_{*1}X_2 = (\mu_{g_1} q L_u R_{u^{-1}})_{*1}X_2 \\ &= (\mu_{g_1} q I_u)_{*1}X_2 = (\mu_{g_1})_{*o}q_{*1}Ad_u X_2\end{aligned}$$

so that $Ad_u X_2 = X_1$ and we conclude that $u \cdot (g_2, X_2) = (g_1, X_1)$. □

1.3 Sprays and connections

In this section we recall some facts about sprays and connections, see Capter IV Sections 3 and 4 in [36] for further information. A *second-order vector field* on a manifold M is a vector field $F : TM \to TTM$ on TM satisfying $\pi_* \circ F = id_{TM}$, where $\pi : TM \to M$ is the projection map ([36] IV, 3). Let $s \in \mathbb{R}$ and let $s_{TM} : TM \to TM$, $X \mapsto sX$ denote the multiplication by s in each tangent space. A second order vector field is called a *spray* if

$$F(sX) = s_{TM*}(sF(X))$$

for all $s \in \mathbb{R}$ and $X \in TM$. For $X \in T_xM$ let $\gamma_X : J \to TM$ be the maximal integral curve of F with initial condition X, that is $\gamma_X(0) = X$ and

$$\dot{\beta}_X = F(\beta_X).$$

The domain $\mathcal{D}_{exp} \subseteq TM$ is the set of all the vectors $X \in T_xM$ for which the maximal integral curve β_X is defined in $[0, 1]$. The *exponential map* derived from the spray is defined as

$$exp : \mathcal{D}_{exp} \to M, \qquad exp(X) = \pi(\gamma_X(1))$$

and for $x \in M$ we denote by exp_x the restriction of exp to $\mathcal{D}_{exp} \cap T_xM$. The *geodesic* with initial speed $X \in T_xM$ is given by

$$\alpha(t) = \pi(\beta_X(t)).$$

Locally, if U is an open subset of a Banach space V then $TU \simeq U \times V$, $TTU \simeq (U \times V) \times (V \times V)$ and $\pi_{*(x,X)}(Y, Z) = (x, Y)$. A second-order vector field $F : TU \to TTU$ can be written as

$$F(x, X) = (a, X, X, f(x, X))$$

where $f : U \times V \to V$ is a smooth map. The spray condition means that

$$\begin{aligned}(a, sX, sX, f(s, sX)) &= F(x, sX) = (s_{TM})_*(sF(X)) \\ &= s_{TM*}(x, X, sX, sf(X, X)) \\ &= (x, sX, sX, s^2 f(x, X))\end{aligned}$$

which means that the maps $f(x, \cdot)$ are quadratic. Using the polarization identity we can define the *Cristoffel symbols*

$$\Gamma_x(X, Y) = \frac{1}{4}(f(x, X+Y) - f(x, X-Y)), \qquad \text{for } x \in U \text{ and } X, Y \in V.$$

We can locally define a *covariant derivative* as

$$(D_\xi \zeta)_x = \zeta'_x \xi_x - \Gamma_x(\xi_x, \zeta_x).$$

The covariant derivative is a bilinear function

$$\mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M), \quad (\xi, \zeta) \mapsto D_\xi \zeta$$

wich is $C^\infty(M)$ linear in the first variable, i.e. $D_{f\xi}\zeta = fD_\xi\zeta$ for $\xi, \zeta \in \mathcal{V}(M)$, $f \in C^\infty(M)$ and satisfies the Leibniz rule in the second variable, i.e.

$$D_\xi(f\zeta) = \xi(f) + fD_\xi\zeta$$

for $\xi, \zeta \in \mathcal{V}(M)$ and $f \in C^\infty(M)$. Let $\alpha : J \to M$ be a C^2-curve and let $t_0, t_1 \in J$. We denote by

$$P_{t_0}^{t_1}(\alpha) : T_{\alpha(t_0)}M \to T_{\alpha(t_1)}M$$

the corresponding linear map given by *parallel transport* along α.

In a local chart U parallel transport is defined as follows. If $\alpha : J \to U$ is a C^2-curve and $t_0, t_1 \in J$, then for each $v \in T_{\alpha(t_0)}U = V$ let $(\alpha, \gamma_X) : J \to TU = U \times V$ be the unique lift of α with initial condition $\gamma'_X(t_0) = X$ and which is α-parallel, i.e. which solves the first-order linear differential equation

$$\gamma'_X(t) = \Gamma_{\alpha(t)}(\alpha'(t), \gamma_X(t))$$

for all $t \in J$. Then $P_{t_0}^{t_1}(\alpha) : V \to V$ is the linear map defined as $X \mapsto \gamma_X(t_1)$.

1.4 Symmetric spaces

Definition 1.4.1. *Let M be a Banach manifold. We say that (M,μ) is a **symmetric space** in the sence of Loos if*

$$\mu : M \times M \to M, \quad (x,y) \mapsto x \cdot y$$

is a smooth map with the following properties:

S1 *$x \cdot x = x$ for all $x \in M$.*

S2 *$x \cdot (x \cdot y) = y$ for all $x, y \in M$.*

S3 *$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$ for all $x, y, z \in M$.*

S4 *Every $x \in M$ has a neighborhood U such that $x \cdot y = y$ implies $x = y$ for all $y \in U$, hence x is an isolated fixed point of the morphism $y \to x \cdot y$ for all $x \in M$.*

See [40] where this axioms where defined for finite dimensional manifolds.

For $x \in M$ we define a map $\sigma : M \to M$, $\sigma_x(y) = x \cdot y$. For all $x \in M$ this map satisfies

$$(\sigma_x)_{*x} = -id_{T_xM}.$$

This follows from the fact that σ_x is an involution with isolated fixed point x, see S2 and S4. If we identify $T(M \times M)$ with $TM \times TM$ then

$$X \cdot X = \mu_{*(x,y)}(X,Y)$$

for $X \in T_xM$ and $Y \in T_yM$ defines on TM the structure of a symmetric space. In each tangent space T_xM the product satisfies $X \cdot Y = 2X - Y$. For $X \in TM$ we write $\sigma_X = TM \to TM$ for the symmetry given by $\sigma_X(Y) = \mu_*(X,Y) = X \cdot Y$ and $\mathcal{O} : M \to TM$ the zero section. The function

$$F : TM \to TTM, \qquad F(X) = -(\sigma_{\frac{X}{2}} \circ \mathcal{O})_* X$$

defines a spray on M, see Theorem 3.4 in [48]. Note that $\sigma_{\frac{X}{2}} \circ \mathcal{O} : M \to TM$ so that $(\sigma_{\frac{X}{2}} \circ \mathcal{O})_* : TM \to TTM$.

If $\alpha : \mathbb{R} \to M$ is a geodesic then we call the maps $\tau_{\alpha,s} = \sigma_{\alpha(\frac{s}{2})} \circ \sigma_{\alpha(0)}$, $s \in \mathbb{R}$, *translations along* α. The following is Theorem 3.6 in [48].

Theorem 1.4.2. *Let (M,μ) be a connected symmetric space and F the canonical spray defined based on μ. Then*

- *$Aut(M,\mu) = Aut(M,F)$, i.e. a diffeomorphism ϕ of M satisfies $F \circ \phi_* = (\phi_*)_* \circ F$ if and only if $\phi \circ \mu = \mu \circ (\phi \times \phi)$.*
- *F is uniquely determined by the property of being invariant under all symmetries $\{\sigma_x\}_{x \in M}$.*
- *(M,F) is geodesically complete, i.e. $\mathcal{D}_{exp} = TM$.*
- *Let $\alpha : \mathbb{R} \to M$ be a geodesic and let $\tau_{\alpha,s} = \sigma_{\alpha(\frac{s}{2})} \circ \sigma_{\alpha(0)}$, $s \in \mathbb{R}$ be the translations along α. Then these are automorphisms of (M,μ) with*

$$\tau_{\alpha,s}(\alpha(t)) = \alpha(t+s)$$

and parallel transport along the geodesic α is given by

$$(\tau_{\alpha,s})_{*\alpha(t)} = P_t^{t+s}(\alpha) : T_{\alpha(t)}M \to T_{\alpha(t+s)}M$$

for $s,t \in \mathbb{R}$.

1.4.1 Banach Lie groups with involution

A connected Lie group G with an involutive automorphism σ is called a *symmetric Banach-Lie group.* Let $\mathfrak{g}$ be the Banach-Lie algebra of G, and let

$$U = \{g \in G : \sigma(g) = g\}$$

be the subgroup of fixed points of σ. Then the Banach-Lie algebra $\mathfrak{u}$ of U is a closed and complemented subspace of $\mathfrak{g}$, a complement is given by the closed subspace

$$\mathfrak{p} = \{X \in \mathfrak{g} : \sigma_{*1}X = -X\}.$$

The Lie algebra $\mathfrak{u}$ is the eigenspace of σ_{*1} corresponding to the eigenvalue $+1$ and $\mathfrak{p}$ is the eigenspace corresponding to the eigenvalue -1. Since $\mathfrak{u}$ is complemented U is a Banach-Lie subgroup of G, and the quotient space $M = G/U$ has a Banach manifold structure. A natural chart around $o = q(1)$ is given by

$$X \mapsto q(exp(X))$$

restricted to a suitable neighborhood of 0 in $\mathfrak{p}$. Note that $\sigma(e^X) = e^{\sigma_{*1}X} = e^{-X}$ for every $X \in \mathfrak{p}$.

We also define $G^+ = \{g\sigma(g)^{-1} : g \in G\}$, which is a submanifold of G and note that there is a diffeomorphism

$$\phi : G/U \to G^+, \quad gU \mapsto g\sigma(g)^{-1}.$$

We use the notation $g^* = \sigma(g)^{-1}$ for $g \in G$.

There is a smooth action of G on G^+ defined by

$$\psi : G \to Aut(G^+), \quad \psi_x(y) = xyx^* = xy\sigma(x)^{-1}$$

and onother smooth action of G on G/U given by *translation*

$$\tau : G \to Aut(G/U), \quad \tau_g(hU) = ghU.$$

Under the isomorphism ϕ the translation τ_x corresponds to ψ_x, i.e. $\phi \circ \tau_x = \psi_x \circ \phi$ for all $x \in G$. We can define a map

$$\rho : G^+ \times G^+ \to G^+, \quad x \times y = \rho(x, y) = xy^{-1}x$$

and a map

$$\mu : G/U \times G/U \to G/U, \quad gU \times hU = \rho(gU, hU) = g\sigma(g)^{-1}\sigma(h)U.$$

Under the diffeomorphism ϕ the map ρ corresponds to μ, i.e.

$$\phi \circ \mu = \rho \circ (\phi \times \phi).$$

See the Chapter XIII Section 5 in [36] for further information about symmetric spaces.

Proposition 1.4.3. *The action of G on $M = G/U$ is by automorphisms of (M, μ).*

Proof. This follows from

$$\begin{aligned} \mu(g \cdot h_1U, g \cdot h_2U) &= \mu(gh_1U, gh_2U) = gh_1\sigma(gh_1)^{-1}\sigma(gh_2)U \\ &= gh_1\sigma(h_1)^{-1}\sigma(g)^{-1}\sigma(g)\sigma(h_2)U = gh_1\sigma(h_1)^{-1}\sigma(h_2)U \\ &= g \cdot \mu(h_1, h_2). \end{aligned}$$

□

Proposition 1.4.4. *The multiplications on $G^+ \simeq G/U$ satisfy the properties stated in Definition 1.4.1, so (G^+, ρ) and $(G/U, \mu)$ are symmetric spaces and ϕ is an isomorphism of symmetric spaces.*

Proof. We verify (S1), (S2) and (S3) for the multiplication ρ in G^+ and (S4) for the multiplication μ in G/U:

S1 $xx^{-1}x = x$ for $x \in G^+$.

S2 $x(xy^{-1}x)^{-1}x = y$ for $x, y \in G^+$.

S3 $x(yz^{-1}y)^{-1}x = xy^{-1}zy^{-1}x = (xy^{-1}x)(x^{-1}zx^{-1})(xy^{-1}x) = (xy^{-1}x)(xz^{-1}x)^{-1}(xy^{-1}x)$ for $x, y, z \in G^+$.

S4 Since G acts transitively on G/U, it is sufficient to verify this condition in the base-point o. Since $\sigma_o(gU) = o \cdot gU = \sigma(g)U$ we see that $(\sigma_o)_{*o} = -id_{T_oM}$.

□

1.4.2 Geodesics and exponential map of G/U

In this subsection we compute the geodesics, the exponential map and the derivative of the exponential map of a symmetric space $(G/U, \mu)$ derived from a symmetric Banach-Lie group (G, σ).

Proposition 1.4.5. *The geodesics in $(G/U, \mu)$ through $o = q(1)$ are given by*

$$\mathbb{R} \to G/U, \qquad t \mapsto q(exp(tX))$$

with $X \in \mathfrak{p}$.

Proof. We calculate the geodesic α such that $\alpha(0) = q(1) = o$ and $\dot{\alpha}(0) = Y = q_{*o}X \in T_oM$ for an $X \in \mathfrak{p}$ by computing the flows of Killing fields in two different ways. Killing fields are infinitessimal automorphisms of symmetric spaces, i.e. if $(\phi_t)_{t\in\mathbb{R}}$ is a one-parameter family of automorphisms of (M, μ) then

$$\chi = \frac{d}{dt}\Big|_{t=0} \phi_t : M \to TM$$

is a Killing field. Killing fields satisfying $\chi(o) = Y = q_{*o}X \in T_oM$ and which are parallel at o, i.e. $(\sigma_o)_* \circ \chi = -\chi \circ \sigma_o$, are unique.

If $\alpha : \mathbb{R} \to M$ is the geodesic with $\alpha(0) = o$ and $\dot{\alpha}(0) = Y$ let $\tau_t = \sigma_{\alpha(\frac{t}{2})} \circ \sigma_{\alpha(0)}$ denote the translations along α which form a one-parameter group of automorphisms. Then

$$\xi_Y = \frac{d}{dt}\Big|_{t=0} \tau_t : M \to TM$$

is a Killing vector field on M wich satisfies $\xi_Y(0) = Y$. Since $\sigma_x\sigma_y\sigma_x = \sigma_{\sigma_x y}$ in every symmetric space we see that

$$\sigma_o\tau_t = \sigma_o\sigma_{\alpha(\frac{t}{2})}\sigma_o = \sigma_{\sigma_o\alpha(\frac{t}{2})} = \sigma_{\alpha(-\frac{t}{2})} = \sigma_{\alpha(-\frac{t}{2})}\sigma_o\sigma_o = \tau_{-t}\sigma_o.$$

If we derivate the last equality in $t = 0$ we get

$$(\sigma_o)_* \circ \xi_Y = -\xi_Y \circ \sigma_o.$$

Consider the one-parameter group on M given by

$$\lambda_t(gU) = exp(tX)gU.$$

Since this one-parameter group is by automorphisms of (M, μ) its infinitesimal vector field

$$\eta_X(gU) = \frac{d}{dt}\Big|_{t=0} exp(tX)gU$$

is a Killing vector field which satisfies $\eta_X(o) = q_{*1}X$. Since $X \in \mathfrak{p}$ we see that

$$\sigma_o(exp(tX)gU) = o \cdot exp(tX)gU = \sigma(exp(tX)g)U = exp(-tX)\sigma(g)U$$

so that differentiating this equation at $t = 0$ we get

$$(\sigma_o)_* \circ \eta_X(gU) = \frac{d}{dt}\Big|_{t=0} exp(-tX)\sigma(g)U = -\frac{d}{dt}\Big|_{t=0} exp(tX)\sigma(g)U.$$

Also

$$-\eta_X \circ \sigma_o(gU) = -\eta_X(o \cdot gU) = -\eta_X(\sigma(g)U) = -\frac{d}{dt}\Big|_{t=0} exp(tX)\sigma(g)U.$$

and we get

$$(\sigma_o)_* \circ \eta_X = -\eta_X \circ \sigma_o.$$

By uniqueness we conclude that $\xi_Y = \eta_X$ and that the flows of these two vector fields are equal, so that the geodesic α is given by

$$\alpha(t) = \tau_t(o) = \lambda_t(o) = exp(tX)U = q(exp(tX)).$$

□

Therefore the exponential at o of the symmetric space $M = G/U$ is given by

$$exp_o(q_{*1}X) = q(exp(X)).$$

If define $Exp = q \circ exp$ we have $Exp = exp_o \circ q_{*1}$.

Corollary 1.4.6. *Since the action of G on $M = G/U$ is transitive geodesics through $q(g)$ are given by*

$$\mathbb{R} \to G/U, \qquad t \mapsto q(gexp(tX))$$

with $X \in \mathfrak{p}$.

Remark 1.4.7. *We note that $\sigma(I_u e^{tX}) = I_u e^{-tX}$ for every $X \in \mathfrak{p}$ and $u \in U$, so that $\sigma_{*1} Ad_u X = -Ad_u X$ and $\mathfrak{p}$ is Ad_U-invariant. Since σ is a group automorphism, σ_{*1} is an automorphism of Lie algebras and the following inclusions hold:*

$$[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{u}, \quad [\mathfrak{u}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{u}.$$

In particular, $\mathfrak{p}$ is $ad_{\mathfrak{u}}$-invariant.

For each $X \in \mathfrak{g}$ we have by [34, Theorem IV.4.1]

$$exp_{*X} = (L_{exp(X)})_{*1} \frac{1 - e^{-ad_X}}{ad_X}.$$

Using this equation we can derive the formula for the differential of the exponential map.

Proposition 1.4.8. *If Exp is the exponential map of the symmetric space G/U then*

$$Exp_{*X} = (\mu_{exp(X)})_{*o} \frac{\sinh ad_X}{ad_X}\Big|_{\mathfrak{p}}$$

for $X \in \mathfrak{p} \simeq T_o M$.

Proof. Since $Exp = q \circ exp$, differentiating at $X \in \mathfrak{p}$ we get

$$Exp_{*X} = q_{*exp(X)} exp_{*X} = q_{*exp(X)} (L_{exp(X)})_{*1} \frac{1 - e^{-ad_X}}{ad_X}.$$

Differentiating $\mu_{exp(X)} \circ q = q \circ L_{exp(X)}$ at 1 yields

$$q_{*exp(X)} (L_{exp(X)})_{*1} = (\mu_{exp(X)})_{*o} q_{*1}$$

so that

$$Exp_{*X} = (\mu_{exp(X)})_{*o} q_{*1} \frac{1 - e^{-ad_X}}{ad_X}.$$

Writing $\frac{1-e^{-ad_X}}{ad_X}$ in a power series and then as a sum of the even and odd powers we see that

$$\frac{1 - e^{-ad_X}}{ad_X} = \frac{1 - \cosh ad_X}{ad_X} + \frac{\sinh ad_X}{ad_X}.$$

Using the properties of the bracket in Remark 1.4.7 we see that for $Y \in \mathfrak{p}$

$$\frac{1 - \cosh ad_X}{ad_X} Y \in \mathfrak{u} \text{ and } \frac{\sinh ad_X}{ad_X} Y \in \mathfrak{p}$$

so the stated formula follows. ☐

Corollary 1.4.9. *The map Exp_{*X} is invertible if and only if $Spec((ad_X)^2|_{\mathfrak{p}}) \cap \{-n^2\pi^2 : n \in \mathbb{N}\} = \{0\}$.*

1.4.3 Connection, geodesics and exponential of P

If A is a unital C^*-algebra, G is the group of invertible elements of A endowed with the manifold structure given by the norm and σ is given by

$$\sigma : G \to G, \quad g \mapsto (g^{-1})^*,$$

then $U = \{g \in G : \sigma(g) = g\}$ is the group of unitary operators of A. In this case $\mathfrak{p} = A_s$ the set of self-adjoint elements of A and $\mathfrak{u} = A_{as}$ is the set of skew-adjoint elements of A. We have an ismorphism $G/U \simeq G^+$, $gU \mapsto gg^*$ where $G^+ = \{gg^* : g \in G\}$ is the set of positive invertible elements in A which we will denote by P. Since P is an open subset of the real Banach space A_s of self-adjoint elements of A it is a submanifold of the manifold A_s.

Also, P has the structure of symmetric space with symmetries given by $\mu(a,b) = \sigma_a(b) = a \cdot b = ab^{-1}a$ for $a, b \in P$. The associated symmetries in TP are

$$(a, X) \cdot (b, Y) = (ab^{-1}a, Xb^{-1}a + ab^{-1}X - ab^{-1}Yb^{-1}a)$$

which restricted to T_aP are

$$(a, X) \cdot (a, Y) = (a, 2X - Y).$$

Therefore $\sigma_{(a,\frac{1}{2}X)} \circ Z : P \to TP$ is given by

$$b \mapsto (b, 0) \mapsto (a, \frac{1}{2}X) \cdot (b, 0) = (ab^{-1}a, \frac{1}{2}(Xb^{-1}a + ab^{-1}X))$$

so that $(\sigma_{(a,\frac{1}{2}X)} \circ Z)_{*b}(Y) =$

$$(ab^{-1}a, \frac{1}{2}(Xb^{-1}a + ab^{-1}X), -ab^{-1}Yb^{-1}a, -\frac{1}{2}(Xb^{-1}Yb^{-1}a + ab^{-1}Yb^{-1}X).$$

Hence

$$F(a, X) = -(\sigma_{(a,\frac{1}{2}X)} \circ Z)_{*(a,X)} = (a, X, X, Xa^{-1}X).$$

We see that in this case $f(a, X) = Xa^{-1}X$ so that by polarization the Cristoffel symbol is

$$\Gamma_a(X, Y) = \frac{1}{2}(Xa^{-1}Y + Ya^{-1}X).$$

Proposition 1.4.10. *The exponential map at* 1 *is given by*

$$exp_1(1, X) = e^X$$

for $X \in A_s$, hence the geodesics through 1 *with initial speed $X \in A_s \simeq T_1P$ is given by*

$$\mathbb{R} \to P, \qquad t \mapsto e^{tX}$$

Proof. Let $\gamma_{(1,X)} : \mathbb{R} \to TP$ be the unique smooth curve in TP such that

$$\gamma_{(1,X)}(0) = (1, X), \text{ and } (\gamma_{(1,X)})\dot{}(t) = F(\gamma_{(1,X)}(t)).$$

Such curve is given by

$$\gamma_{(1,X)}(t) = (e^{tX}, Xe^{tX})$$

since $\gamma_{(1,X)}(0) = (1, X)$ and

$$(\gamma_{(1,X)})\dot{}(t) = (e^{tX}, Xe^{tX}, Xe^{tX}, X^2e^{tX}) = F(e^{tX}, Xe^{tX}) = F(\gamma_{(1,X)}(t)).$$

The exponential map at 1 is given by

$$exp_1(1, X) = \pi(\gamma_{(1,X)}(1)) = \pi(e^X, Xe^X) = e^X$$

which is the usual exponential. We note that the geodesic $\alpha : \mathbb{R} \to P$ such that $\alpha(0) = 1$ and $\dot{\alpha}(0) = X$ is given by

$$\alpha(t) = \pi(\gamma_{(1,X)}(t)) = \pi((e^{tX}, Xe^{tX})) = e^{tX}.$$

□

Corollary 1.4.11. *Since the connection is invariant under the transitive action of G on P, if γ is a geodesic and $g \in G$ it follows that $\psi_g \circ \gamma$ is a geodesic. Therefore*

$$\begin{array}{ccc} P & \xrightarrow{\psi_{a^{\frac{1}{2}}}} & P \\ {\scriptstyle exp_1}\uparrow & & \uparrow{\scriptstyle exp_a} \\ T_1P & \xrightarrow{(\psi_{a^{\frac{1}{2}}})_{*1}} & T_aP \end{array}$$

or

$$a^{\frac{1}{2}}exp(X)a^{\frac{1}{2}} = exp_a(a^{\frac{1}{2}}Xa^{\frac{1}{2}})$$

for $a \in P$ and $X \in A_s \simeq T_1P$, so that the exponential map of the connection of P at $a \in P$ is given by

$$exp_a : T_aP \to P, \qquad exp_a(X) = a^{\frac{1}{2}}e^{a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}}a^{\frac{1}{2}} = ae^{a^{-1}X} = e^{Xa^{-1}}a$$

for $a \in P$ and $X \in A_s \simeq T_aP$.

Therefore the unique geodesic γ such that $\gamma(0) = a$ and $\dot{\gamma}(0) = X \in A_s \simeq T_aP$ is

$$\gamma(t) = a^{\frac{1}{2}}e^{ta^{-\frac{1}{2}}Xa^{-\frac{1}{2}}}a^{\frac{1}{2}}.$$

If we use the function $log : P \to A_s$ which is the inverse of $exp : A_s \to P$ obtained applying the analytic functional calculus we can compute the unique geodesic $\gamma_{a,b} : [0, 1] \to P$ joining a and b. It is given by

$$\gamma_{a,b}(t) = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^t a^{\frac{1}{2}},$$

where $a^t = e^{t.log(a)}$ as usual.

1.5 Finsler structure and distance

1.5.1 Definitions

The following is Definition 1.3 and Definition 1.4 in [48].

Definition 1.5.1. *Let M be a Banach manifold. A **tangent norm** on M is a function $b : TM \to \mathbb{R}^+$ whose restriction to every tangent space T_xM is a norm. We write $\|X\|_x = b_x(X)$ for $X \in T_xM$ and $x \in M$. A continuous tangent norm b on M is called **compatible** if for each $x \in M$ there exists a chart $\phi : U \to V$ around x and constants $m, M > 0$ such that*

$$m.b(X) \leq \|\phi_{*x}X\| \leq M.b(X)$$

*for all $x \in U$ and $v \in T_xM$. A **Finsler manifold** is a pair (M, b) of a Banach manifold M and a compatible tangent norm b. A metric d on M is called a **locally compatible metric** if for each $x \in M$ there exists a chart $\phi : U \to V$ around x and constants $m, M > 0$ such that*

$$m.d(x, y) \leq \|\phi(x) - \phi(y)\| \leq M.d(x, y)$$

for all $x, y \in U$.

Definition 1.5.2. *A metric d is called a **compatible metric** if it is locally compatible and the topology induced from the metric d coincides with the original topology. A **metric Banach manifold** is a pair (M, d) of a Banach manifold M and a compatible metric d. If (M, b) is a Finsler manifold we define the **length** $Length(\gamma)$ of a piecewise C^1-curve $\gamma : J \to M$ as the improper Riemann integral*

$$Length(\gamma) = \int_J b_{\gamma(t)}(\dot{\gamma}(t))dt \in [0, \infty] = \int_J \|\dot{\gamma}(t)\|_{\gamma(t)}dt \in [0, \infty].$$

We obtain a metric d on M by

$$d(x, y) = inf\{Length(\gamma) : \gamma \text{ is a piecewise smooth curve joining } x \text{ and } y\}.$$

*We call (M, b) **complete** if it is a complete metric space with respect to the metric d.*

The following is Proposition 12.22 in [63] which implies that every Finsler manifold is a metric Banach manifold in a canonical fashion.

Proposition 1.5.3. *The metric d on a Finsler manifold (M, b) is compatible and invariant under the group of all diffeomorphisms ϕ of M with $b \circ \phi_* = b$.*

Definition 1.5.4. *Let (M,μ) be a connected symmetric space, F the canonical spray on M, and b a compatible tangent norm on M. If b is invariant under parallel transport, then we call (M,b,F) a* ***Finsler manifold with spray****.*

The following is Corollary 1.11 in [48].

Proposition 1.5.5. *If (M,b,F) is a complete Finsler manifold with spray then (M,d) is complete if and only if M is geodesically complete, i.e. $\mathcal{D}_{exp} = TM$.*

Note that in the finite dimensional theory of Finsler manifolds the function $b : TM \to \mathbb{R}^+$ is assumed to be smooth on the complement of the zero section and strictly convex on each tangent space. In our infinite dimensional context we do not assume these conditions.

1.5.2 Finsler structure on G/U

If (G,σ) is a symmetric Banach-Lie group we want to turn $M = G/U$ into a Finsler manifold on which G acts isometrically, see the paragraph previous to Lemma 3.10 in [48]. We assume that there is a norm on $\mathfrak{p}$ compatible with norm on T_oM given by any local chart wich is invariant under the group $Ad(U)$, i.e. $\|Ad_u(X)\| = \|X\|$ for every $u \in U$ and $X \in \mathfrak{p}$. We identify $TM \simeq G \times_U \mathfrak{p}$ as in Theorem 1.2.13. Then $b : TM \simeq G \times_U \mathfrak{p} \to \mathbb{R}^+$, $b([g,X]) = \|X\|$ is well defined and defines a tangent norm on M.

Proposition 1.5.6. *The tangent norm given by $b : TM \to \mathbb{R}^+$ is compatible with the topology of M.*

Proof. The function
$$F : \mathfrak{p} \to \mathcal{B}(\mathfrak{p}), \quad X \mapsto \frac{\sinh ad_X}{ad_X}$$
is continuous with $F(0) = 1$, so that there is a neighborhood U of 0 in $\mathfrak{p}$ with $\|F(x)^{-1}\| \leq m$ and $\|F(x)\| \leq M$ for all $x \in U$. Then
$$\frac{1}{m}\|X\| \leq \|Exp_{*x}X\| = \|F(x)X\| \leq M\|X\|$$
for all $x \in U$ and $X \in \mathfrak{p}$. □

The following proposition is evident.

Proposition 1.5.7. *The function $b : TM \to \mathbb{R}^+$ is invariant under the action of G on $G \times_U \mathfrak{p} \simeq TM$ given by $g \cdot [h,X] = [gh,X]$, or alternatively by $g \cdot X = (\mu_g)_* X$.*

Proposition 1.5.8. *The function $b : TM \to \mathbb{R}^+$ is invariant under parallel transport so that $(G/U,b,F)$ is a Finsler manifold with spray.*

Proof. Parallel transport is the derivative at a point of the composition of two symmetries by the last item in Theorem 1.4.2. Since the composition of two symmetries is a translation and by Proposition 1.5.7 the derivative of translations leaves the function b invariant the statement of the proposition follows. □

To make clear the dependence of M with its underlying Banach-Lie group, involution and Finsler structure we shall write $M = G/U = Sym(G, \sigma, \|\cdot\|_{\mathfrak{p}})$ and we shall call M a *Finsler symmetric space.*

1.5.3 Finsler structure on P

Let A is a unital C^*-algebra and $P \simeq G/U$ is the symmetric space described in Subsection 1.4.3. In this case $\mathfrak{p} = A_s$ the set of self-adjoint elements of A and the uniform norm on A_s, which we denote by $\|\cdot\|$, is Ad_U-invariant because it is unitarily invariant. We can identify the manifold G/U with the manifold of positive invertible elements P. If $\phi : G/U \to P$, $gU \mapsto gg^*$ then the identification

$$G \times_U A_s \xrightarrow{\simeq} T(G/U) \xrightarrow{\phi_*} TP$$

$$[g, W] \mapsto q_{*g}(L_g)_{*1}W \mapsto (gg^*, gWg^* + gW^*g^*)$$

implies that a Finsler metric can be defined on P with the norms $\|\cdot\|_a : T_aP \to \mathbb{R}^+$ for $a \in P$ which satisfy

$$\|(\psi_g)_{*a}X\|_{\psi_g(a)} = \|gXg^*\|_{gag^*} = \|X\|_a$$

for every $X \in A_s \simeq T_aP$, $a \in P$ and $g \in G$. Then, for $a \in P$

$$\|(\psi_{a^{\frac{1}{2}}})_{*1}X\|_{\psi_{a^{\frac{1}{2}}}(1)} = \|a^{\frac{1}{2}}Xa^{\frac{1}{2}}\|_a = \|X\|_1 = \|X\|$$

so that

$$\|X\|_a = \|a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\| \text{ for } X \in A_s \simeq T_aP.$$

In this way a Finsler symmetric space $P = G/U = Sym(G, \sigma, \|\cdot\|_{A_s})$ is defined. See [22], where this norm was first introduced. Also note that Proposition 1.5.3 in this case can be formulated as:

Proposition 1.5.9. *The action ψ of G on (P, d) given by $g \cdot a = gag^*$ is isometric.*

See Proposition 1 in [21].

1.6 Symmetric spaces of semi-negative curvature

1.6.1 A generalized Cartan-Hadamard theorem

Definition 1.6.1. *We say that a complete Finsler manifold with spray (M,b,F) has **semi-negative curvature** if for all $x \in M$ and $X \in T_xM$ the operator between Banach spaces*

$$(exp_x)_{*X} : T_x(M) \simeq T_X(T_x(M)) \to T_{exp_x(X)}(M)$$

is expansive and surjective. This means that for $X \in T_xM \cap \mathcal{D}_{exp}$ and $Y \in T_X(T_xM)$

$$\|(exp_x)_{*X}(Y)\| \geq \|Y\|$$

and $(exp_x)_{*X}$ is invertible for each $X \in T_xM \cap \mathcal{D}_{exp}$.

This is Definition 1.4 in [48].

Theorem 1.6.2. *Let (M,b,F) be a connected Finsler manifold with spray which has semi-negative curvature. Then (M,b,F) is complete if and only if it is geodesically complete, and in this case for each $x \in M$ the exponential map*

$$exp_x : T_xM \to M$$

is a surjective covering. In particular, if M is simply connected, then the exponential map $exp_x : T_xM \to M$ is a diffeomorphism.

Definition 1.6.3. *A simply connected complete Finsler manifold with spray (M,b,F) which has semi-negative curvature is called a **Cartan-Hadamard manifold**.*

Remark 1.6.4. *Let (M,b,F) be a Cartan-Hadamard manifold and $x \in M$. If $\Gamma : [0,1] \to T_xM$ is a smooth curve and $\gamma = exp_x \circ \Gamma : [0,1] \to M$, then*

$$Length_{T_xM}(\Gamma) \leq Length_M(\gamma)$$

since

$$\|\dot\gamma(t)\|_{\gamma(t)} = \|(exp_x)_{*\Gamma(t)}(\dot\Gamma(t))\|_{exp_x(\Gamma(t))} \geq \|\dot\Gamma(t)\|_x$$

for $t \in [0,1]$.

Using the inequality in the last remark on can prove the following ***exponential metric increasing property***:

Proposition 1.6.5. *Let (M, b, F) be a Cartan-Hadamard manifold, then for $x \in M$ and $X, Y \in T_xM$*

$$\|X - Y\|_x \leq d(exp_x(X), exp_x(Y))$$

and

$$\|X\|_x = d(x, exp_x(X)).$$

For two points $x, y \in M$ the unique geodesic segment $\alpha_{x,y} : [0, 1] \to M$ joining them is lenght minimizing.

See Theorem 3.5 in Chapter XI, Section 5 of [36] for a proof of this fact in the context of Hilbert manifolds and Lemma 3.1 in [15] for a proof in the present context. The next theorem was proved in [38]

Proposition 1.6.6. *Let (M, b, F) be a Cartan-Hadamard manifold, then for two geodesic segments $\alpha, \beta : [0, 1] \to M$ the distance map*

$$[0, 1] \to [0, +\infty), \quad t \mapsto d(\alpha(t), \beta(t))$$

is convex.

1.6.2 Criterion for semi-negative curvature of G/U

In [48] Neeb established a criterion for semi-negative curvature of a Finsler symmetric space $G/U = Sym(G, \sigma, \|\cdot\|_{\mathfrak{p}})$ using the concepts of dissipative and expansive operator.

Definition 1.6.7. *Let Z be a Banach space. For $z \in Z$ we put*

$$F(z) = \{\alpha \in Z' : \|\alpha\| \leq 1, \langle \alpha, z \rangle = \|z\|\}.$$

*We call $A \in \mathcal{B}(Z)$ **dissipative** if for each $z \in Z$ there exists an $\alpha \in F(z)$ with $Re\langle \alpha, A(z) \rangle \leq 0$.*

The following is Theorem 2.2 in [48].

Theorem 1.6.8. *For $A \in \mathcal{B}(Z)$ the following are equivalent*

- *A is dissipative.*
- *For each $t > 0$ the operator $1 - tA$ is expansive.*
- *$\|e^{tA}\| \leq 1$ holds for all $t > 0$.*

- *$Re\langle \alpha, A(z)\rangle \leq 0$ holds for all $z \in Z$ and $\alpha \in F(z)$.*
- *For each $t > 0$ the operator $1 - tA$ is expansive and suryective.*

Using this theorem Neeb proved a criterion for semi-negative curvature for Finsler symmetric spaces, see [48, Proposition 3.15]:

Theorem 1.6.9. *Let $M = G/U = Sym(G, \sigma, \|\cdot\|_{\mathfrak{p}})$ be a Finsler symmetric space. Then the following conditions are equivalent:*

- *M has semi-negative curvature.*
- *The operator $-(ad_X)^2|_{\mathfrak{p}}$ is dissipative for all $X \in \mathfrak{p}$.*
- *The operator $1 + (ad_X)^2|_{\mathfrak{p}}$ is expansive and invertible for all $X \in \mathfrak{p}$.*
- *The operator $X \in \mathfrak{p}$, $\frac{\sinh ad_X}{ad_X}|_{\mathfrak{p}}$ is expansive and invertible for all $X \in \mathfrak{p}$.*

1.6.3 semi-negative curvature of P

In [20, Theorem 1] the "exponential metric increasing property" which states that for $a \in P$ and $X, Y \in T_aP$

$$\|X - Y\|_a \leq d(exp_a(X), exp_a(Y))$$

was shown to be equivalent to Segal's inequality which states that for self-adjoint operators X and Y

$$\|e^{X+Y}\| \leq \|e^{\frac{X}{2}} e^Y e^{\frac{X}{2}}\|.$$

The following was proved by Corach, Porta and Recht in the Remark at the end of [20] using the exponential metric increasing property

Theorem 1.6.10. *The Finsler symmetric space $P = G/U = Sym(G, \sigma, \|\cdot\|)$ of positive invertible elements of a C^*-algebra has semi-negative curvature.*

Therefore, by Proposition 1.6.6 for two geodesics α and β in P the map

$$[0,1] \to P \qquad t \mapsto d(\alpha(t), \beta(t))$$

is convex. This was proved in Theorem 2 of [23] using [23, Theorem 1] which states that if J is a Jacobi field along a geodesic α in P then

$$t \mapsto \|J(t)\|_{\alpha(t)}$$

is convex. In [3] this fact was shown to be equivalent to the Löwner-Heinz inequality which states that for positive operators A, B and $t \in [0,1]$

$$\|A^t B^t\| \leq \|AB\|^t.$$

In Theorem 6.3 in [22] and Proposition 2. in [21] the following was proved:

Proposition 1.6.11. *The unique geodesic $\gamma_{a,b} : [0,1] \to P$ joining a and b minimizes the distance, which is given by*

$$d(a,b) = Length(\gamma_{a,b}) = \|log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|.$$

Remark 1.6.12. *The geodesic is not unique with this property due to the fact that the norms on the tangent spaces are not uniformly convex.*

In [16] this work was extended to the context of unititzed p-Schatten operators. Let $A = \mathcal{B}(\mathcal{H})$ stand for the set of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$, with the uniform norm denoted by $\|\cdot\|$. For $1 \leq p < \infty$ let A_p be the ideal of p-Schatten operators with p-norm $\|\cdot\|_p$. Let G_p stand for the group of invertible operators in the unitized ideal, that is $G_p = \{g \in A^\times : \ g - 1 \in A_p\}$, then G_p is a Banach-Lie group (one of the so-called classical Banach-Lie groups [33]), and A_p identifies with its Banach-Lie algebra. Consider the involutive automorphism $\sigma : G_p \to G_p$ given by $g \mapsto (g^*)^{-1}$. Let $U_p \subseteq G_p$ stand for the unitary subgroup of fixed points of σ. In this case $\mathfrak{p}$ is the set of self-adjoint operators in A_p and the norm $\|\cdot\|_p$ on $\mathfrak{p}$ is Ad_{U_p}-invariant. We can identify the manifold G_p/U_p with the manifold of positive invertible operators in G_p. The following was proved by Conde and Larotonda in the appendix of [16]

Theorem 1.6.13. *The Finsler symmetric space $M_p = G_p/U_p = Sym(G_p, \sigma, \|\cdot\|_p)$ is simply connected and has semi-negative curvature.*

Chapter 2

Decompositions and complexifications of some infinite-dimensional homogeneous spaces

2.1 Introduction

In this chapter we extend certain results on the geometric description of complexifications of homogeneous spaces of Banach-Lie groups studied by Beltiţă and Galé in [6] and also the decompositions of the acting groups by means of a series of chained reductive structures.

In Section 2.2 we recall the definition of reductive structures, which can be interpreted as connection forms E on homogeneous spaces of the form G_A/G_B. Examples in the context of operator algebras are given: conditional expectations, their restrictions to Schatten ideals and projections to corners of operator algebras. The Corach-Porta-Recht splitting theorem by Conde and Larotonda [16] is used to prove an extended Corach-Porta-Recht splitting theorem in the context of several reductive structures.

In Section 2.3 the Corach-Porta-Recht splitting theorem is used to give a geometric description of homogeneous spaces of the form G_A/G_B as associated principal bundles over U_A/U_B. Under additional hypothesis about the holomorphic character of G_A and the involution σ on G_A it is possible to interpret G_A/G_B as the complexification of U_A/U_B. Under these additional assumptions G_A/G_B is identified with the tangent bundle of U_A/U_B and it is shown that this identification has nice functorial properties related to the connection form E. Finally, we use the three examples of connection forms introduced in Section 2.2, to give a geometrical description of the complexifications of flag manifolds, coadjoint orbits in Schatten ideals and Stiefel manifolds respectively.

2.2 Splitting of Finsler symmetric spaces

2.2.1 Polar and Corach-Porta-Recht decomposition

We recall some facts about the fundamental group of M and polar decompositions [48, Theorem 3.14 and Theorem 5.1] which are consequences of the Cartan-Hadamard theorem 1.6.2.

Theorem 2.2.1. *Let $M = G/U = Sym(G, \sigma, \| \cdot \|_{\mathfrak{p}})$ be a Finsler symmetric space of semi-negative curvature, then*

1. *The exponential map $q \circ Exp : \mathfrak{p} \to M$ is a covering of Banach manifolds and*
$$\Gamma = \{X \in \mathfrak{p} : q(e^X) = q(1)\}$$
is a discrete and additive subgroup of $\mathfrak{p} \cap Z(\mathfrak{g})$, with $\Gamma \simeq \pi_1(M)$ and $M \simeq \mathfrak{p}/\Gamma$. $Z(\mathfrak{g})$ denotes the center of the Banach-Lie algebra $\mathfrak{g}$. If $X, Y \in \mathfrak{p}$ and $q(e^X) = q(e^Y)$, then $X - Y \in \Gamma$.

2. *The polar map*
$$m : \mathfrak{p} \times U \to G, \quad (X, u) \mapsto e^X u$$
is a surjective covering whose fibers are given by the sets $\{(X - Z, e^Z u) : \; Z \in \Gamma\}$, $u \in U$, $X \in \mathfrak{p}$. If M is simply connected the map m is a diffeomorphism.

Let A be a unital C^*-algebra, since by Theorem 1.6.10 G/U is simply connected and has semi-negative curvature we get the usual polar decomposition of invertible elements as a product of a positive invertible element and a unitary.

Corollary 2.2.2. *In the context of Theorem 2.2.1 $G_A^+ = e^{\mathfrak{p}}$. Note that given $h \in G_A^+$ there is a $g \in G_A$ such that $h = g\sigma(g)^{-1}$. Using the polar decomposition in G_A there are $X \in \mathfrak{p}$ and $u \in U$ such that $g = e^X u$. Then $h = e^X u \sigma(e^X u)^{-1} = e^X u u^{-1} e^X = e^{2X} \in e^{\mathfrak{p}}$. We note also that $e^X = e^{\frac{1}{2}X} \sigma(e^{\frac{1}{2}X})^{-1} \in G_A^+$ for every $X \in \mathfrak{p}$.*

The following decomposition theorem in the context of Finsler symmetric spaces of semi-negative curvature was proven by Conde and Larotonda in [16].

Theorem 2.2.3. *Corach-Porta-Recht decomposition*

Let $M = G/U = (G, \sigma, \| \cdot \|_{\mathfrak{p}})$ be a simply connected Finsler symmetric space of semi-negative curvature. Let $p \in \mathcal{B}(\mathfrak{p})$ be an idempotent, $p^2 = p$. Let $\mathfrak{s} := Ran(p)$, $\mathfrak{s}' := Ran(1-p) = Ker(p)$, so that $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}'$. If $ad_{\mathfrak{s}}^2(\mathfrak{s}) \subseteq \mathfrak{s}$, $ad_{\mathfrak{s}}^2(\mathfrak{s}') \subseteq \mathfrak{s}'$ and $\|p\| = 1$, then the maps
$$\Phi : U \times \mathfrak{s}' \times \mathfrak{s} \to G, \qquad (u, X, Y) \mapsto u e^X e^Y$$

$$\Psi : \mathfrak{s}' \times \mathfrak{s} \to G^+, \qquad (X, Y) \mapsto e^Y e^{2X} e^Y$$

are diffeomorphisms.

2.2.2 Reductive structures with involution

The following two definitions are from Beltiţă and Galé [7].

Definition 2.2.4. *A **reductive structure** is a triple $(G_A, G_B; E)$ where G_A is a real or complex connected Banach-Lie group with Banach-Lie algebra $\mathfrak{g}_A$, G_B is a connected Banach-Lie subgroup of G_A with Banach-Lie algebra $\mathfrak{g}_B$, and $E : \mathfrak{g}_A \to \mathfrak{g}_A$ is a $\mathbb{R}$-linear continuous transformation which satisfies the following properties: $E \circ E = E$; $RanE = \mathfrak{g}_B$, and for every $g \in G_B$ the diagram*

$$\begin{array}{ccc} \mathfrak{g}_A & \xrightarrow{E} & \mathfrak{g}_B \\ {\scriptstyle Ad_g}\downarrow & & \downarrow{\scriptstyle Ad_g} \\ \mathfrak{g}_A & \xrightarrow{E} & \mathfrak{g}_B \end{array}$$

commutes.

Definition 2.2.5. *A **morphism of reductive structures** from $(G_A, G_B; E)$ to $(\tilde{G_A}, \tilde{G_B}; \tilde{E})$ is a homomorphism of Banach-Lie groups $\alpha : G_A \to \tilde{G_A}$ such that $\alpha(G_B) \subseteq \tilde{G_B}$ and such that the diagram*

$$\begin{array}{ccc} \mathfrak{g}_A & \xrightarrow{E} & \mathfrak{g}_B \\ {\scriptstyle \alpha_{*1}}\downarrow & & \downarrow{\scriptstyle \alpha_{*1}} \\ \tilde{\mathfrak{g}_A} & \xrightarrow{\tilde{E}} & \tilde{\mathfrak{g}_B} \end{array}$$

commutes.

For example, a family of automorphisms of any reductive structure $(G_A, G_B; E)$ is given by $I_g : x \mapsto gxg^{-1}$, $G_A \to G_A$, $(g \in G_B)$.

Remark 2.2.6. *If $(G_A, G_B; E)$ is a reductive structure, the the identity map $id_{G_A} : G_a \to G_A$ is a morphism of reductive structures because $id_{G_A}(G_B) = G_B$ and $(id_{G_A})_{*1} \circ E = E \circ (id_{G_A})_{*1}$. If α is a morphism of reductive structures from $(G_A, G_B; E)$ to $(\tilde{G_A}, \tilde{G_B}; \tilde{E})$ and β is a morphism of reductive structures from $(\tilde{G_A}, \tilde{G_B}; \tilde{E})$ to $(\hat{G_A}, \hat{G_B}; \hat{E})$ then*

$$\beta \circ \alpha(G_B) = \beta(\alpha(G_B)) \subseteq \beta(\tilde{G_B}) \subseteq \hat{G_B}$$

and

$$(\beta \circ \alpha)_{*1} \circ E = \beta_{*1} \circ \alpha_{*1} \circ E = \beta_{*1} \circ \tilde{E} \circ \alpha_{*1} = \hat{E} \circ \beta_{*1} \circ \alpha_{*1} = \hat{E} \circ (\beta \circ \alpha)_{*1}$$

so that $\beta \circ \alpha$ is a morphism of reductive structures from $(G_A, G_B; E)$ to $(\hat{G_A}, \hat{G_B}; \hat{E})$. We conclude that we can define a category whose objects are reductive structures and whose morphism are morphisms of reductive structures.

Now we introduce involutions in reductive structures:

Definition 2.2.7. *If $(G_A, G_B; E)$ is a reductive structure and σ is an involutive morphism of reductive structures we call $(G_A, G_B; E, \sigma)$ a* ***reductive structure with involution****. If $(G_A, G_B; E, \sigma)$ and $(\tilde{G_A}, \tilde{G_B}; \tilde{E}, \tilde{\sigma})$ are reductive structures with involution and α is a morphism of reductive structures from $(G_A, G_B; E)$ to $(\tilde{G_A}, \tilde{G_B}; \tilde{E})$ such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$ then we call α a* ***morphism of reductive structures with involution*** *from $(G_A, G_B; E, \sigma)$ to $(\tilde{G_A}, \tilde{G_B}; \tilde{E}, \tilde{\sigma})$.*

As in Remark 2.2.6 the reductive structures with involution and morphisms of reductive structures with involution are a category.

Definition 2.2.8. *If B is a C^*-subalgebra of a C^*-algebra A then a $\mathbb{C}$-linear projection $E : A \to A$ with $RanE = B$, $E(1_A) = 1_B(= 1_A)$ and $\|E\| = 1$ is called a* ***conditional expectation****. By Tomiyama's theorem [62] the following holds*

$$E(b_1 a b_2) = b_1 E(a) b_2 \qquad \text{for all } a \in A; \quad b_1, b_2 \in B$$

$$E(a^*) = E(a)^* \qquad \text{for all } a \in A.$$

Example 2.2.9. *Conditional expectations in C^*-algebras*

Let A and B be two unital C^*-algebras, such that B is a subalgebra of A and let $E : A \to B$ be a conditional expectation. Let G_Λ for $\Lambda \in \{A, B\}$ be the Banach-Lie group of invertible operators in Λ endowed with the topology given by the uniform norm. Then the Banach-Lie algebra of G_Λ is $\mathfrak{g}_\Lambda = \Lambda$. Since in this case we have $Ad_g(X) = gXg^{-1}$ for each $g \in G_A$ and $X \in A \simeq T_1G$ we see that

$$E(Ad_g(X)) = E(gXg^{-1}) = gE(X)g^{-1} = Ad_g(E(X))$$

for $g \in G_B$ and $X \in A \simeq T_1G$ so the expectation $E : \mathfrak{g}_A = A \to \mathfrak{g}_B = B$ satisfies the conditions of Def. 2.2.4. We conclude that $(G_A, G_B; E)$ is a reductive structure. In fact, this is a classical example that was the motivation of that definition in the paper [7].

If $(G_A, G_B; E)$ is a reductive structure that is derived from an inclusion of C^*-algebras and a conditional expectation as above then $\sigma : G_A \to G_A$, $a \mapsto (a^{-1})^*$ is involutive, satisfies $\sigma(G_B) = G_B$ and since $\sigma_{*1} : A \to A$, $X \mapsto -X^*$ it also satisfies

$$E(\sigma_{*1}(X)) = E(-X^*) = -E(X)^* = \sigma_{*1}(E(X))$$

for $X \in A$. Hence σ defines an involutive automorphism of reductive structures and $(G_A, G_B; E, \sigma)$ is a reductive structure with involution.

If for two triples $(A, B; E)$, $(\tilde{A}, \tilde{B}; \tilde{E})$ there is a bounded $*$-homomorphism $\phi : A \to \tilde{A}$ which satisfies $\phi \circ E = \tilde{E} \circ \phi$, then $\alpha = \phi|_{G_A}$ defines a morphism of reductive structures with involution from $(G_A, G_B; E, \sigma)$ to $(\tilde{G_A}, \tilde{G_B}; \tilde{E}, \tilde{\sigma})$. To see this note that from $\phi \circ E = \tilde{E} \circ \phi$ it follows that $\phi(B) = \phi(E(A)) = \tilde{E}(\phi(A)) \subseteq \tilde{E}(\tilde{A}) = \tilde{B}$ so that $\alpha(G_B) \subseteq \tilde{G_B}$. Note that $\alpha_{*1} = \phi : A = \mathfrak{g}_A \to \tilde{A} = \tilde{\mathfrak{g}_A}$ so that $\alpha_{*1} \circ E = \tilde{E} \circ \alpha_{*1}$. Also $\alpha(\sigma(a)) = \alpha((a^{-1})^*) = ((\alpha(a))^{-1})^* = \tilde{\sigma}(\alpha(a))$ for $a \in G_A$.

Example 2.2.10. *We use the notation of the last paragraph of Subsection 1.6.3 where groups of Schatten perturbations of the identity is discussed. Let $B \subseteq A = \mathcal{B}(\mathcal{H})$ be a C^*-subalgebra, and let $E : A \to B$ be a conditional expectation with range B such that E sends trace-class operators to trace-class operators and E is compatible with the trace, that is $Tr(E(x)) = Tr(x)$ for any trace-class operator $x \in A$. Let $p \geq 1$, A_p be the ideal of p-Schatten operators in A, $B_p = B \cap A_p$,*

$$G_{A,p} = \{g \in A^\times : g - 1 \in A_p\} \;\; \text{and} \;\; G_{B,p} = \{g \in A^\times : g - 1 \in B_p\}.$$

Then $\mathfrak{g}_{A,p} = A_p$ and $\mathfrak{g}_{B,p} = B_p$ are the Banach-Lie algebras of $G_{A,p}$ and $G_{B,p}$ respectively. It was proven in Section 5 of [16] that $E_p = E|_{A_p} : A_p \to B_p$ and that $\|E_p\| = 1$. It easy to see that $(G_{A,p}, G_{B,p}; E_p, \sigma)$ is a reductive structure with involution.

Example 2.2.11. *Corners*

Let $\mathcal{H}$ be a Hilbert space, $n \geq 1$ and p_i, $i = 1, \ldots, n+1$ be pairwise orthogonal non-zero projections with range $\mathcal{H}_i$ and $\sum_{i=1}^{n-1} p_i = 1$. Let G_A be the group of invertible elements of $\mathcal{B}(\mathcal{H})$ and let

$$G_B = \left\{ \begin{pmatrix} g_1 & 0 & \ldots & 0 & 0 \\ 0 & g_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & g_n & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} : g_i \text{ invertible in } \mathcal{B}(\mathcal{H}_i) \text{ for } i = 1, \ldots, n \right\};$$

where we write operators in $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_{n+1})$ as $(n+1) \times (n+1)$ matrices with the corresponding operator entries.

In this case $\mathfrak{g}_A = \mathcal{B}(\mathcal{H})$ and

$$\mathfrak{g}_B = \left\{ \begin{pmatrix} X_1 & 0 & \ldots & 0 & 0 \\ 0 & X_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & X_n & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} : X_i \text{ in } \mathcal{B}(\mathcal{H}_i) \text{ for } i = 1, \ldots, n \right\}.$$

If we consider the map $E:\mathfrak{g}_A\to\mathfrak{g}_B$, $X\mapsto\sum_{i=1}^n p_iXp_i$ and $\sigma=(\cdot)^{*-1}$ it is easily verified that $(G_A,G_B;E,\sigma)$ is a reductive structure with involution. To see this note that if

$$g=\begin{pmatrix} g_1 & 0 & \dots & 0 & 0\\ 0 & g_2 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & g_n & 0\\ 0 & 0 & \dots & 0 & 1\end{pmatrix}\in G_B$$

and

$$X=\begin{pmatrix} X_{1,1} & X_{1,2} & \dots & X_{1,n} & X_{1,n+1}\\ X_{2,1} & X_{2,2} & \dots & X_{2,n} & X_{2,n+1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ X_{n,1} & X_{n,2} & \dots & X_{n,n} & X_{n,n+1}\\ X_{n+1,1} & X_{n+1,2} & \dots & X_{n+1,n} & X_{n+1,n+1}\end{pmatrix}\in\mathcal{B}(\mathcal{H})=T_1G_A,$$

then

$$Ad_gX=gXg^{-1}=\begin{pmatrix} g_1X_{1,1}g_1^{-1} & g_1X_{1,2}g_2^{-1} & \dots & g_1X_{1,n}g_n^{-1} & g_1X_{1,n+1}\\ g_2X_{2,1}g_1^{-1} & g_2^X{}_{2,2}g_2^{-1} & \dots & g_2X_{2,n}g_n^{-1} & g_2X_{2,n+1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ g_nX_{n,1}g_1^{-1} & g_nX_{n,2}g_2^{-1} & \dots & g_nX_{n,n}g_n^{-1} & g_nX_{n,n+1}\\ X_{n+1,1}g_1^{-1} & X_{n+1,2}g_2^{-1} & \dots & X_{n+1,n}g_n^{-1} & X_{n+1,n+1}\end{pmatrix},$$

so that

$$E(Ad_gX)=\begin{pmatrix} g_1^{-1}X_{1,1}g_1^{-1} & 0 & \dots & 0 & 0\\ 0 & g_2^{-1}X_{2,2}g_2^{-1} & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & g_n^{-1}X_{n,n}g_n^{-1} & 0\\ 0 & 0 & \dots & 0 & 0\end{pmatrix}.$$

Aslo note that $\|E\|=1$ since we get E by first taking the diagonal blocks and then making the last block of the diagonal blocks zero and these two operators have norm 1.

2.2.3 Extended Corach-Porta-Recht decomposition

Definition 2.2.12. *If (G_A,σ) is a symmetric Banach-Lie group we say that a connected subgroup $G_B\subseteq G_A$ is **involutive** if $\sigma(G_B)=G_B$.*

The next lemma is Corollary II.3 in [48].

Lemma 2.2.13. *If $A \in \mathcal{B}(Z)$ is a bounded operator on the Banach space Z and W is a A-invariant subspace of Z, then $A|_W$ is dissipative.*

Remark 2.2.14. *If $G_B \subseteq G_A$ is an involutive Banach-Lie subgroup with Banach-Lie algebra $\mathfrak{g}_B \subseteq \mathfrak{g}_A$ and $\mathfrak{g}_A = \mathfrak{p} \oplus \mathfrak{u}$ is the eigenspace decomposition of σ_{*1}, we can write $\mathfrak{g}_B = \mathfrak{p}_B \oplus \mathfrak{u}_B$, where $\mathfrak{p}_B := \mathfrak{p} \cap \mathfrak{g}_B$ and $\mathfrak{u}_B := \mathfrak{u} \cap \mathfrak{g}_B$.*

Proposition 2.2.15. *Given a Finsler symmetric space*

$$M_A = G_A/U_A = Sym(G_A, \sigma, \| \cdot \|_{\mathfrak{p}})$$

of semi-negative curvature, if G_B is an involutive subgroup, then

$$M_B = G_B/U_B = Sym(G_B, \sigma|_{G_B}, \| \cdot \|_{\mathfrak{p}_B})$$

is a Finsler symmetric space of semi-negative curvature. Also, by adapting the notation of the first item of Theorem 2.2.1, the inclusion $\Gamma_B \subseteq \Gamma_A \cap \mathfrak{p}_B$ holds. In particular, if M_A is simply connected then M_B is also simply connected.

Proof. We can restrict the Ad_{U_A}-invariant norm of $M_A = G_A/U_A$ to $\mathfrak{p}_B$ to give $M_B = G_B/U_B$ a Ad_{U_B}-invariant norm. Since for each $X \in \mathfrak{p}$ the operator $-(ad_X)^2|_{\mathfrak{p}}$ is dissipative and $-(ad_X)^2|_{\mathfrak{p}}(\mathfrak{p}_B) \subseteq \mathfrak{p}_B$ for all $X \in \mathfrak{p}_B$, we conclude by Lemma 2.2.13 that the operator $-(ad_X)^2|_{\mathfrak{p}_B}$ is dissipative for all $X \in \mathfrak{p}_B$. Therefore $M_B = G_B/U_B = Sym(G_B, \sigma|_{G_B}, \| \cdot \|_{\mathfrak{p}_B})$ has semi-negative curvature.

If $X \in \Gamma_B$ then $q_B \circ Exp_B(X) = o_B$ so that $Exp_A(X) = Exp_B(X) \in U_B \subseteq U_A$ and $q_A \circ Exp_A = o_A$. We conclude that $\Gamma_B \subseteq \Gamma_A \cap \mathfrak{p}_B$. □

Remark 2.2.16. *If $(G_A, G_B; E)$ is a reductive structure, since $Ad_g \circ E = E \circ Ad_g$ for each $g \in G_B$ we see that $Ad_g(KerE) \subseteq KerE$ for every $g \in G_B$. If σ is an involutive automorphism of reductive structures and $\mathfrak{g}_A = \mathfrak{u} \oplus \mathfrak{p}$ is the decomposition into eigenspaces of σ_{*1} then $Ad_{U_A}(\mathfrak{p}) \subseteq \mathfrak{p}$ and $Ad_{U_A}(\mathfrak{u}) \subseteq \mathfrak{u}$, so that the actions $Ad : U_B \to \mathcal{B}(\mathfrak{p}_E)$ and $Ad : U_B \to \mathcal{B}(\mathfrak{u}_E)$ are well defined, where we denote $\mathfrak{p}_E := KerE \cap \mathfrak{p}$ and $\mathfrak{u}_E := KerE \cap \mathfrak{u}$.*

The following theorem is a generalization of the Corach-Porta-Recht-type decomposition from Theorem 2.2.3.

Theorem 2.2.17. *If for $n \geq 2$ we have the following inclusions of connected Banach-Lie groups, the following maps between their Lie algebras*

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n$$

$$\mathfrak{g}_1 \overset{E_2}{\longleftarrow} \mathfrak{g}_2 \overset{E_3}{\longleftarrow} \ldots \overset{E_n}{\longleftarrow} \mathfrak{g}_n$$

and a morphism $\sigma : G_n \to G_n$ such that:

- $(G_n, G_{n-1}; E_n, \sigma)$,$(G_{n-1}, G_{n-2}; E_n, \sigma|_{G_{n-1}})$,..., $(G_2, G_1; E_2, \sigma|_{G_2})$ *are reductive structures with involution.*
- $M_n = G_n/U_n = Sym(G_n, \sigma, \|\cdot\|)$ *is a simply connected Finsler symmetric space of semi-negative curvature.*
- $\|E_k|_{\mathfrak{p}_k}\| = 1$ *for* $k = 2, \ldots, n$, *where we use the norm of the previous item restricted to* $\mathfrak{p}_k := \mathfrak{p} \cap \mathfrak{g}_k$.

Then the maps

$$\Phi_n : U_n \times \mathfrak{p}_{E_n} \times \cdots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1 \to G_n$$
$$(u_n, X_n, \ldots, X_2, Y_1) \mapsto u_n e^{X_n} \ldots e^{X_2} e^{Y_1}$$

$$\Psi_n : \mathfrak{p}_{E_n} \times \cdots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1 \to G_n^+$$
$$(X_n, \ldots, X_2, Y_1) \mapsto e^{Y_1} e^{X_2} \ldots e^{X_{n-1}} e^{2X_n} e^{X_{n-1}} \ldots e^{X_2} e^{Y_1}$$

are diffeomorphisms, where $\mathfrak{p}_{E_k} := KerE_k \cap \mathfrak{p}_k$ *for* $k = 2, \ldots, n$.

Proof. Note that Prop. 2.2.15 implies that $M_k := G_k/U_k$ are simply connected Finsler symmetric spaces of semi-negative curvature for $k = 2, \ldots, n$. We prove the statement about the map Φ for the case $n = 2$ and then prove the statement for $n > 2$ by induction.

Since $E_2 \circ \sigma_{*1} = \sigma_{*1} \circ E_2$, $E_2(\mathfrak{p}_2) \subseteq \mathfrak{p}_2$, we can consider $p := E_2|_{\mathfrak{p}_2} : \mathfrak{p}_2 \to \mathfrak{p}_2$. We see that $\|p\| = 1$ and $Ker(p) = Ran(1-p) = \mathfrak{p}_{E_2}$. Also, since $E_2^2 = E_2$ and $Ran(E_2) = \mathfrak{g}_1$, $Ran(p) = \mathfrak{p}_1$. The condition $ad^2_{\mathfrak{p}_1}(\mathfrak{p}_1) \subseteq \mathfrak{p}_1$ of the statement of Theorem 2.2.3 is trivial. Also note that for every $g \in G_1$ and for every $X \in \mathfrak{g}_2$, $Ad_g(E_2(X)) = E_2(Ad_g(X))$. If $Y \in \mathfrak{g}_1$ and we differentiate $Ad_{e^{tY}}(E_2(X)) = E_2(Ad_{e^{tY}}(X))$ at $t = 0$ we get $ad_Y(E_2(X)) = E_2(ad_Y(X))$ and therefore $ad_{\mathfrak{g}_1}(KerE_2) \subseteq KerE_2$. Since $ad^2_{\mathfrak{p}_2}(\mathfrak{p}_2) \subseteq \mathfrak{p}_2$ we conclude that $ad^2_{\mathfrak{p}_1}(\mathfrak{p}_{E_2}) \subseteq \mathfrak{p}_{E_2}$. Theorem 2.2.3 implies the existence of a diffeomorphism

$$\Phi_2 : U_2 \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1 \to G_2$$
$$(u_2, X_2, Y_1) \mapsto u_2 e^{X_2} e^{Y_1}. \tag{2.1}$$

Assume now that $n > 2$ and that the theorem is true for $k = n-1$ and $k = 2$. We prove that Φ_n is surjective. If $g_n \in G_n$ then the splitting (2.1) derived above from Theorem 2.2.3 applied to the reductive structure $(G_n, G_{n-1}; E_n)$ implies the existence of $u_n \in U_n$, $X_n \in \mathfrak{p}_{E_n}$ and Y_{n-1} such that $g_n = u_n e^{X_n} e^{Y_{n-1}}$. Since $e^{Y_{n-1}} \in G_{n-1}$ applying (2.1) in the case $k = n-1$ we get $u_{n-1} \in U_{n-1}$, $X_{n-1} \in \mathfrak{p}_{E_{n-1}}$,..., $X_2 \in \mathfrak{p}_{E_2}$ and $Y_1 \in \mathfrak{p}_1$ such that $e^{Y_{n-1}} = u_{n-1} e^{X_{n-1}} \ldots e^{X_2} e^{Y_1}$. Then

$$g_n = u_n e^{X_n} e^{Y_{n-1}} = u_n e^{X_n} u_{n-1} e^{X_{n-1}} \ldots e^{X_2} e^{Y_1} = u_n u_{n-1} e^{Ad_{u_{n-1}^{-1}} X_n} e^{X_{n-1}} \ldots e^{X_2} e^{Y_1}$$

is in the image of Φ_n because $Ad_{u_{n-1}^{-1}} X_n \in \mathfrak{p}_{E_n}$.

We prove that Φ_n is injective. Assume that

$$u_n e^{X_n} e^{X_{n-1}} \ldots e^{X_2} e^{Y_1} = u_n' e^{X_n'} e^{X_{n-1}'} \ldots e^{X_2'} e^{Y_1'}.$$

Since $e^{X_{n-1}} \ldots e^{X_2} e^{Y_1} \in G_{n-1}$ there are $u_{n-1} \in U_{n-1}$ and $Y_{n-1} \in \mathfrak{p}_{n-1}$ such that

$$u_{n-1} e^{Y_{n-1}} = e^{X_{n-1}} \ldots e^{X_2} e^{Y_1}.$$

Also there are $u_{n-1}' \in U_{n-1}$ and $Y_{n-1}' \in \mathfrak{p}_{n-1}$ such that

$$u_{n-1}' e^{Y_{n-1}'} = e^{X_{n-1}'} \ldots e^{X_2'} e^{Y_1'}.$$

Then

$$u_n u_{n-1} e^{Ad_{u_{n-1}}^{-1} X_n} e^{Y_{n-1}} = u_n' u_{n-1}' e^{Ad_{u'^{-1}_{n-1}} X_n'} e^{Y_{n-1}'}$$

and because of the uniqueness of the splitting theorem for $k = 2$ we conclude that

$$\begin{aligned} u_n u_{n-1} &= u_n' u_{n-1}' \\ Ad_{u_{n-1}^{-1}} X_n &= Ad_{u'^{-1}_{n-1}} X_n' \\ Y_{n-1} &= Y_{n-1}'. \end{aligned} \tag{2.2}$$

Since $u_{n-1} e^{Y_{n-1}} = e^{X_{n-1}} \ldots e^{X_2} e^{Y_1}$ and $u_{n-1}' e^{Y_{n-1}'} = e^{X_{n-1}'} \ldots e^{X_2'} e^{Y_1'}$

$$u_{n-1}^{-1} e^{X_{n-1}} \ldots e^{X_2} e^{Y_1} = e^{Y_{n-1}} = e^{Y_{n-1}'} = u'^{-1}_{n-1} e^{X_{n-1}'} \ldots e^{X_2'} e^{Y_1'}$$

the uniqueness of the splitting theorem for $k = n - 1$ implies that $u_{n-1} = u_{n-1}'$, $X_{n-1} = X_{n-1}'$,..., $X_2 = X_2'$ and $Y_1 = Y_1'$. The equalities in (2.2) say that $u_n = u_n'$ and $X_n = X_n'$ also hold.

We prove that Ψ_n is bijective based on the fact that Φ_n is bijective. If $p_n \in G_A^+$ then $p_n = g_n g_n^*$ for some $g_n \in G_n$. Because Φ_n is surjective there are $u_n \in U_n$, $X_n \in \mathfrak{p}_{E_n}, \ldots,$ $X_2 \in \mathfrak{p}_{E_2}$ and $Y_1 \in \mathfrak{p}_1$ such that $g_n^* = u_n e^{X_n} \ldots e^{X_2} e^{Y_1}$. Then $p_n = g_n g_n^* = e^{Y_1} e^{X_2} \ldots e^{2X_n} \ldots e^{X_2} e^{Y_1}$ and we conclude that Ψ_n is surjective. To see that Ψ_n is injective let assume that $e^{Y_1} e^{X_2} \ldots e^{2X_n} \ldots e^{X_2} e^{Y_1} = e^{Y_1'} e^{X_2'} \ldots e^{2X_n'} \ldots e^{X_2'} e^{Y_1'}$. If $g_n := e^{Y_1} e^{X_2} \ldots e^{X_n}$ and $g'_n := e^{Y_1'} e^{X_2'} \ldots e^{X_n'}$ then $g_n g_n^* = g'_n g'^*_n$ and therefore there is an $u_n \in U_n$ such that $g_n u_n = g_n'$. Then $u_n e^{X_n} \ldots e^{X_2} e^{Y_1} = e^{X_n'} \ldots e^{X_2'} e^{Y_1'}$ and we conclude that $(X_n, \ldots, X_2, Y_1) = (X_n', \ldots, X_2', Y_1')$.

We prove that Φ_n is a diffeomorphism by induction. Theorem 2.2.3 states that Φ_2 is a diffeomorphism. Assume that $n > 2$ and that Φ_{n-1} is a diffeomorphism. If $g_n \in G_n$ then $g_n = u_n(g_n) e^{X_n(g_n)} e^{Y_{n-1}(g_n)}$, where $(u_n, X_n, Y_{n-1}) : G_n \to U_n \times \mathfrak{p}_{E_n} \times \mathfrak{p}_{n-1}$ is smooth because the inverse of the Corach-Porta-Recht splitting is smooth in the case $n = 2$. If we denote $f(g_n) := e^{Yn-1(g_n)}$ then f is a smooth map and

$$f(g_n) = u_{n-1}(f(g_n))e^{X_{n-1}(f(g_n))}\dots e^{X_2(f(g_n))}e^{Y_1(f(g_n))}$$

where

$$(u_{n-1}, X_{n-1}, \dots, X_2, Y_1) : G_{n-1} \to U_{n-1} \times \mathfrak{p}_{E_{n-1}} \times \dots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1$$

is a smooth map. Since

$$g_n = u_n(g_n)e^{X_n(g_n)}u_{n-1}(f(g_n))e^{X_{n-1}(f(g_n))}\dots e^{X_2(f(g_n))}e^{Y_1(f(g_n))} =$$

$$u_n(g_n)u_{n-1}(f(g_n))e^{Ad_{u_{n-1}^{-1}(f(g_n))}X_n(g_n)}e^{X_{n-1}(f(g_n))}\dots e^{X_2(f(g_n))}e^{Y_1(f(g_n))}$$

we get that $\Phi_n^{-1} : G_n \to U_n \times \mathfrak{p}_{E_n} \times \dots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1$

$$g_n \mapsto (u_n(g_n)u_{n-1}(f(g_n)), Ad_{u_{n-1}^{-1}(f(g_n))}X_n(g_n), \dots, X_2(f(g_n)), Y_1(f(g_n)))$$

is smooth.

We prove next that $\Psi^{-1} = (\overline{X_n}, \dots, \overline{X_2}, \overline{Y_1})$ is smooth. Let $g_n \in G_n$, then if $p_n = g_n^* g_n$,

$$p_n = e^{(\overline{Y}_1(p_n))}e^{(\overline{X}_2(p_n))}\dots e^{(\overline{X}_{n-1}(p_n))}e^{(2\overline{X}_n(p_n))}e^{(\overline{X}_{n-1}(p_n))}\dots e^{(\overline{X}_2(p_n))}e^{(\overline{Y}_1(p_n))}.$$

Since $g_n = u_n(g_n)e^{X_n(g_n)}\dots e^{X_2(g_n)}e^{Y_1(g_n)}$ where $\Phi^{-1} = (u_n, X_n, \dots, X_2, Y_1)$, we get

$$p_n := g_n^* g_n = e^{Y_1(g_n)}e^{X_2(g_n)}\dots e^{X_{n-1}(g_n)}e^{2X_n(g_n)}e^{X_{n-1}(g_n)}\dots e^{X_2(g_n)}e^{Y_1(g_n)}$$

so that

$$(\overline{X_n}, \dots, \overline{X_2}, \overline{Y_1}) = (X_n, \dots, X_2, Y_1) \circ \pi$$

where $\pi : G_n \to G_n^+, g_n \to g_n^* g_n$. Since π is a submersion we conclude that $\Psi^{-1} = (\overline{X}_n, \dots, \overline{X}_2, \overline{Y}_1)$ is smooth.

□

Remark 2.2.18. *We note that in the context of the previous theorem, if $F_{k,j} := E_{j+1} \circ \dots \circ E_k$, then $(G_k, G_j; F_{k,j})$ is a reductive structure and $\|F_{k,j}|_{\mathfrak{p}_k}\| = 1$.*

Remark 2.2.19. *The splitting theorem of Porta and Recht [57] asserts that if we have a unital inclusion of C^*-algebras $B \subseteq A$ and a conditional expectation $E : A \to B$ then the map*

$$\Phi : U_A \times \mathfrak{p}_E \times \mathfrak{p}_B \to G_A$$

$$(u, X, Y) \mapsto ue^X e^Y$$

is a diffeomorphism, where $\mathfrak{p}_E$ are the self-adjoint elements of $KerE$ and $\mathfrak{p}_B$ are the self-adjoint elements of B.

Theorem 2.2.17 in the case $n = 2$ is a formulation of the Corach-Porta-Recht splitting (Theorem 2.2.3) in the context of reductive structures. The aformentioned Porta-Recht splitting theorem is a special case of the previous theorem if we consider $(G_A, G_B; E, \sigma)$ derived from the triple $(A, B; E)$ as in Example 2.2.9 and verify that the conditions of the theorem are satisfied because of what was stated in Theorem 1.6.10. The Corach-Porta-Recht theorem covers the case where the inclusion of algebras and the map E are not unital, as in Example 2.2.11 of reductive structures. It also covers the case where the symmetric space and reductive structure are derived from unitized ideals of operators as in 2.2.10, see the appendix in [16].

The Corach-Porta-Recht theorem in the context of several reductive structures (Theorem 2.2.17) covers for example the case of multiple unital inclusions of C^*-algebras and conditional expectations between them

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$$
$$A_1 \xleftarrow{E_2} A_2 \xleftarrow{E_3} \ldots \xleftarrow{E_n} A_n.$$

2.3 Complexifications

2.3.1 Complexifications of homogeneous spaces

Proposition 2.3.7 to Remark 2.3.15 here are extensions of Section 5 of [6], from the context of C^*-algebras to the context of Finsler symmetric spaces of semi-negative curvature with reductive structures.

Definition 2.3.1. *A continuous map $F : X \times [0, 1] \to X$ is called a strong deformation retraction of a space X onto a subspace A if for $x \in X$, $a \in A$ and $t \in [0, 1]$*

$$F(x, 0) = x, \quad F(x, 1) \in A, \quad F(a, t) = a.$$

If such a map F exists then A is a ***strong deformation retract*** *of X.*

Definition 2.3.2. *If U is an open subset of a complex Banach space Z and W is a another complex Banach space then a smooth map $\phi : U \to W$ is called* ***holomorphic*** *if $\phi_{*x} : T_xU = Z \to T_{\phi(x)}W = W$ is $\mathbb{C}$-linear for all $x \in U$, and is called* ***anti-holomorphic*** *if ϕ_{*x} is conjugate linear for all $x \in U$, i.e. $\phi_{*x}(\lambda X) = \bar{\lambda}\phi_{*x}(X)$ for $x \in U$, $X \in Z$ and $\lambda \in \mathbb{C}$. A Banach manifold is a* ***complex Banach manifold*** *if it is modeled on a complex Banach space and it has an atlas such that the transition maps are holomorphic.*

Definition 2.3.3. *Let X be a Banach manifold. A* ***complexification*** *of X is a complex Banach manifold Y endowed with an anti-holomorphic involutive diffeomorphism σ such that the fixed point submanifold $Y_0 = \{y \in Y : \sigma(y) = y\}$ is a strong deformation retract of Y and Y_0 is also diffeomorphic to X.*

Example 2.3.4. *Let $M = G/U = Sym(G, \sigma, \|\cdot\|)$ be a simply connected Finsler symmetric space of semi-negative curvature. Theorem 2.2.1 guarantees that U is a strong deformation retract of G. If G is a complex Banach-Lie group and σ is anti-holomorphic, then G is a complexification of U. In the context of C^*-algebras the group of invertible elements G is a complexification of the group of unitary elements U with $\sigma = (\cdot)^{-1*}$. Note that U is not a complex analytic manifold.*

Definition 2.3.5. *Let (G_A, σ) be a symmetric Banach-Lie group with involutive subgroup G_B. We define $\sigma_G : G_A/G_B \to G_A/G_B$, $uG_B \mapsto \sigma(u)G_B$ and $\lambda : U_A/U_B \hookrightarrow G_A/G_B$, $uU_B \mapsto uG_B$.*

We now give a criterion which implies that U_A/U_B is diffeomorphic to the fixed point set of the involution σ_G.

Proposition 2.3.6. *If $M_A = G_A/U_A = Sym(G_A, \sigma, \|\cdot\|)$ is a Finsler symmetric space of semi-negative curvature, G_B is an involutive subgroup of G_A, and $\Gamma \subseteq \mathfrak{p}_B$, then $G_A^+ \cap G_B = G_B^+$.*

Proof. Since $G_B^+ \subseteq G_A^+ \cap G_B$ always holds, it is enough to prove that $G_A^+ \cap G_B \subseteq G_B^+$. By Corollary 2.2.2 $G_A^+ = e^{\mathfrak{p}}$ and $G_B^+ = e^{\mathfrak{p}_B}$. If $g \in G_A^+ \cap G_B$ then there is an $X \in \mathfrak{p}$ such that $g = e^X$. Since G_B is an involutive subgroup G_B/U_B has semi-negative curvature and using the polar decomposition of Theorem 2.2.1 in G_B guaranties the existence of $u \in U_B$ and $Y \in \mathfrak{p}_B$ such that $g = ue^Y$. Then, Theorem 2.2.1 applied to G_A tells us that for certain $Z \in \Gamma$, $u = e^Z$ and $Y = X - Z$. Since $\Gamma \subseteq \mathfrak{g}_B$ we conclude that $X \in \mathfrak{g}_B$ and therefore $g \in G_B^+$. □

Proposition 2.3.7. *If $G_B^+ = G_A^+ \cap G_B$, then $\lambda(U_A/U_B) = \{s \in G_A/G_B : \sigma_G(s) = s\}$.*

Proof. The inclusion $\subseteq$ is obvious. Given $s = uG_B$ such that $\sigma_G(s) = s$, $u^{-1}\sigma(u) \in G_B$. Since $u^{-1}\sigma(u) \in G_A^+$ the hypothesis $G_B^+ = G_A^+ \cap G_B$ implies that $u^{-1}\sigma(u) \in G_B^+$, and therefore there exists $w \in G_B$ such that $u^{-1}\sigma(u) = ww^*$. Then $uw = \sigma(u)w^{*-1} = \sigma(u)\sigma(w) = \sigma(uw)$, so that $uw \in U_A$ and $s = uG_B = uwG_B = \lambda(uwU_B)$. □

We give a geometric description of the complexification G_A/G_B of U_A/U_B in the context of reductive structures. This can be seen as an infinite dimensional version of Mostow fibration, see [46, 44] and Section 3 of [10].

Remark 2.3.8. *Since the actions $Ad : U_B \to \mathcal{B}(\mathfrak{p}_E)$ and $Ad : U_B \to \mathcal{B}(\mathfrak{u}_E)$ are well defined we get the homogeneous vector bundles $U_A \times_{U_B} \mathfrak{p}_E \to U_A/U_B$ and $U_A \times_{U_B} \mathfrak{u}_E \to U_A/U_B$, $[(u, X)] \mapsto uU_B$, where the actions of U_B on $U_A \times_{U_B} \mathfrak{p}_E$ and $U_A \times_{U_B} \mathfrak{u}_E$ are given by $v \cdot (u, X) = (uv^{-1}, Ad_v X)$.*

Theorem 2.3.9. *Let $M_A = G_A/U_A = Sym(G_A, \sigma, \|\cdot\|)$ be a simply connected Finsler symmetric space of semi-negative curvature and $(G_A, G_B; E, \sigma)$ a reductive structure with involution such that $\|E|_\mathfrak{p}\| = 1$. Consider $\Psi_0^E : U_A \times \mathfrak{p}_E \to G_A$, $(u, X) \mapsto ue^X$ and $\kappa : (u, X) \mapsto [(u, X)]$ the quotient map. Then there is a unique real analytic, U_A-equivariant diffeomorphism $\Psi^E : U_A \times_{U_B} \mathfrak{p}_E \to G_A/G_B$ such that the diagram*

$$\begin{array}{ccc} U_A \times \mathfrak{p}_E & \xrightarrow{\Psi_0^E} & G_A \\ \downarrow \kappa & & \downarrow q \\ U_A \times_{U_B} \mathfrak{p}_E & \xrightarrow{\Psi^E} & G_A/G_B \end{array}$$

commutes.

Therefore the homogeneous space G_A/G_B has the structure of an U_A-equivariant fiber bundle over U_A/U_B with the projection given by the composition

$$G_A/G_B \xrightarrow{(\Psi^E)^{-1}} U_A \times_{U_B} \mathfrak{p}_E \xrightarrow{\Xi} U_A/U_B$$

$$ue^X G_B \mapsto [(u, X)] \mapsto uU_B \qquad \text{for } u \in U_A \text{ and } X \in \mathfrak{p}_E$$

and typical fiber $\mathfrak{p}_E$.

Proof. To prove that Ψ^E is well defined we show that for $u \in U_A$, $v \in U_B$ and $X \in \mathfrak{p}_E$

$$\begin{aligned} q(\Psi_0^E(u, X)) &= ue^X G_B = uv^{-1} e^{Ad_v X} v G_B = uv^{-1} e^{Ad_v X} G_B \\ &= q(\Psi_0^E(uv^{-1}, Ad_v X)) = q(\Psi_0^E(v \cdot (u, X))) \end{aligned}$$

The uniqueness of Ψ^E is a consequence of the surjectivity of κ.

Theorem 2.2.17 for the case $n = 2$ implies the existence of a diffeomorphism

$$\Phi : U_A \times \mathfrak{p}_E \times \mathfrak{p}_B \to G_A$$

$$(u, X, Y) \mapsto ue^X e^Y.$$

If $gG_B \in G_A/G_B$ there is $(u, X, Y) \in U_A \times \mathfrak{p}_E \times \mathfrak{p}_B$ such that $g = ue^X e^Y$ and we get $gG_B = ue^X e^Y G_B = ue^X G_B$, proving the surjectivity of Φ.

To see that Ψ^E is also injective assume that $u_1 e^{X_1} G_B = u_2 e^{X_2} G_B$. Then there is a $b \in G_B$ such that $u_1 e^{X_1} b = u_2 e^{X_2}$. Since G_B is an involutive connected subgroup of G_A

and G_A/U_A has semi-negative curvature, Proposition 2.2.15 states that G_B/U_B has also semi-negative curvature and we can apply the polar decomposition (Proposition 2.2.1) in G_B: there are unique $v \in U_B$ and $Y \in \mathfrak{p}_B$ such that $b = ve^Y$. Then

$$(u_1 v)e^{Ad_{v^{-1}}X_1}e^Y = u_1 e^{X_1} v e^Y = u_1 e^{X_1} b = u_2 e^{X_2}$$

and applying $(\Phi)^{-1}$ to this equality we get $(u_1 v, Ad_{v^{-1}}X_1, Y) = (u_2, X_2, 0)$, which implies that $v^{-1} \cdot (u_1, X_1) = (u_2, X_2)$.

Finally, we prove that Ψ^E is an analytic diffeomorphism. Since κ is a submersion and $\Psi^E \circ \kappa$ $(= q \circ \Psi_0^E)$ is a real analytic map Ψ^E is real analytic. Since the map $\Phi^{-1} : g \mapsto (u(g), X(g), Y(g))$ is analytic, the map $\sigma : g \mapsto [(u(g), X(g))]$, $G_A \to U_A \times_{U_B} \mathfrak{p}_E$ is also analytic. Since q is a submersion and $\sigma = (\Psi^E)^{-1} \circ q$ we see that $(\Psi^E)^{-1}$ is analytic. $\square$

Corollary 2.3.10. *If we analyse the diagram of Theorem 2.3.9 in the tangent spaces using the following identifications $T_{(1,0)}(U_A \times \mathfrak{p}_E) \simeq \mathfrak{u}_A \times \mathfrak{p}_E$, $T_{[(1,0)]}(U_A \times_{U_B} \mathfrak{p}_E) \simeq \mathfrak{u}_E \times \mathfrak{p}_E$ and $T_o(G_A/G_B) \simeq KerE$ then*

$$(\Phi_0^E)_{*(1,0)} : \mathfrak{u}_A \times \mathfrak{p}_E \to \mathfrak{g}_A, \qquad (Y, Z) \mapsto Y + Z$$

$$\kappa_{*(1,0)} : \mathfrak{u}_A \times \mathfrak{p}_E \to \mathfrak{u}_E \times \mathfrak{p}_E, \qquad (Y, Z) \mapsto ((1-E)Y, Z)$$

$$q_{*1} : \mathfrak{g}_A \mapsto KerE, \qquad W \mapsto (1-E)W$$

and therefore

$$(\Phi^E)_{*[(1,0)]} : \mathfrak{u}_E \times \mathfrak{p}_E \to KerE, \qquad ((1-E)Y, Z) \mapsto (1-E)(Y+Z) = (1-E)Y + Z.$$

We conclude that

$$(\Phi^E)_{*[(1,0)]} : \mathfrak{u}_E \times \mathfrak{p}_E \to KerE, \qquad (X, Z) \mapsto X + Z$$

is an isomorphism.

Corollary 2.3.11. *If we assume the conditions of Theorem 2.3.9, the fixed point set of of the involution σ_G on $G_A/G_B \simeq U_A \times_{U_B} \mathfrak{p}_E$ is diffeomorphic to U_A/U_B and U_A/U_B is a strong deformation retract of G_A/G_B. If G_A is a complex Banach-Lie group and σ is anti-holomorphic then G_A/G_B is a complexification of U_A/U_B.*

If we define $\tau_G : U_A \times_{U_B} \mathfrak{p}_E \to U_A \times_{U_B} \mathfrak{p}_E$, $[(u, X)] \mapsto [(u, -X)]$, then the following diagram

$$\begin{array}{ccc} U_A \times_{U_B} \mathfrak{p}_E & \xrightarrow{\tau_G} & U_A \times_{U_B} \mathfrak{p}_E \\ \downarrow{\scriptstyle \Psi^E} & & \downarrow{\scriptstyle \Psi^E} \\ G_A/G_B & \xrightarrow{\sigma_G} & G_A/G_B \end{array}$$

commutes.

Proof. Note that $\Gamma = \{0\}$ so that Prop. 2.3.6 implies $G_B^+ = G_B \cap G_A^+$ and Prop. 2.3.7 states that U_A/U_B is diffeomorphic to the set of fixed points on σ_G.

Alternatively, the diagram tells us that the set of fixed points of the involution σ_G is $\Psi^E(\{[(u,X)] \in U_A \times_{U_B} \mathfrak{p}_E : \tau_G([(u,X)]) = [(u,X)]\}) = \Psi^E(\{[(u,0)] : u \in U_A\}) = \{uG_B : u \in U_A\} = \lambda(U_A/U_B)$.

If we define $F : (U_A \times_{U_B} \mathfrak{p}_E) \times [0,1] \to U_A \times_{U_B} \mathfrak{p}_E, ([(u,X)],t) \mapsto [(u,tX)]$ we see that $\{[(u,0)] : u \in U_A\}$ is a strong deformation retract of $U_A \times_{U_B} \mathfrak{p}_E$ and $\{[(u,0)] : u \in U_A\}$ is diffeomorphic to U_A/U_B.

If σ is anti-holomorphic then σ_G is anti-holomorphic, and it follows from Definition 2.3.3 that G_A/G_B is a complexification of U_A/U_B. □

2.3.2 Complex structure on $T(U_A/U_B)$

Using the Mostow fibration obtained in Theorem 2.3.9 we construct under certain conditions an isomorphism $T(U_A/U_B) \simeq G_A/G_B$ between the tangent space and the complexification of the homogeneous spaces U_A/U_B. This isomorphism gives the tangent spaces $T(U_A/U_B)$ a complex structure.

Theorem 2.3.12. *If we assume that the conditions of Theorem 2.3.9 are satisfied then there is a U_A-equivariant vector bundle isomorphism from the associated vector bundle $U_A \times_{U_B} \mathfrak{u}_E \to U_A/U_B$ onto the tangent bundle $T(U_A/U_B) \to U_A/U_B$ given by $\alpha^E : U_A \times_{U_B} \mathfrak{u}_E \to T(U_A/U_B)$, $[(u,X)] \mapsto (\mu_u)_{*o} q_{*1} X$, where the action of U_A on $T(U_A/U_B)$ is given by $u \cdot - = (\mu_u)_* -$ for every $u \in U_A$.*

Proof. Let $\alpha : U_A \times U_A/U_B \to U_A/U_B$ be given by $(u, vU_B) \mapsto uvU_B$, then $\partial_2\alpha : U_A \times T(U_A/U_B) \to T(U_A/U_B)$, $(u,V) \mapsto (\mu_u)_* V$. Since $E \circ \sigma_{*1} = \sigma_{*1} \circ E$ $E(\mathfrak{u}) \subseteq \mathfrak{u}$, and since $E(\mathfrak{g}_A) = \mathfrak{g}_B$ we get the decomposition $\mathfrak{u} = \mathfrak{u}_B \oplus \mathfrak{u}_E$. Then $\mathfrak{u}_E \simeq T_o(U_A/U_B)$, $X \mapsto q_{*1}X$ and restricting $\partial_2\alpha$ to $U_A \times T_o(U_A/U_B)$ we get a map $\alpha_0^E : U_A \times \mathfrak{u}_E \to T(U_A/U_B)$, $(u,X) \mapsto (\mu_u)_{*o} q_{*1} X$.

As in Theorem 1.2.13 we can prove that there is a unique U_A-equivariant diffeomorphism $\alpha^E : U_A \times_{U_B} \mathfrak{u}_E \to T(U_A/U_B)$ such that $\alpha^E \circ \kappa = \alpha_0^E$, where κ is the quotient map $(u,X) \mapsto [(u,X)]$. □

Lemma 2.3.13. *If σ is an anti-holomorphic involutive automorphism of a complex Banach-Lie group G_A then $i\mathfrak{u} = \mathfrak{p}$.*

Proof. If $X \in \mathfrak{u}$, $\sigma_{*1}X = X$ and $\sigma_{*1}(iX) = -i\sigma_{*1}X = -iX$ so that $iX \in \mathfrak{p}$. The other inclusion is proved in a similar way. □

Example 2.3.14. *If G_A is the group of invertible elements of a C^*-algebra A and σ is the usual involution, then the previous lemma applies and we get $\mathfrak{p} = A_s$ the set of self-adjoint elements of A and $\mathfrak{u} = i\mathfrak{p} = iA_s = A_{as}$ the set of skew-adjoint elements of A.*

Remark 2.3.15. *Assume the conditions of Theorem 2.3.9 are satisfied and that G_A is a complex Banach-Lie group, $\mathfrak{u} = i\mathfrak{p}$, and E is $\mathbb{C}$-linear. Since $Ad_g(iX) = iAd_g(X)$ for every $g \in G_A$ and $X \in \mathfrak{g}_A$ we conclude that $\Theta : U_A \times_{U_B} \mathfrak{p}_E \to U_A \times_{U_B} \mathfrak{u}_E$, given by $[(u,X)] \mapsto [(u,iX)]$ is well defined. Theorem 2.3.9 and Theorem 2.3.12 imply that the composition*

$$G_A/G_B \xrightarrow{(\Psi^E)^{-1}} U_A \times_{U_B} \mathfrak{p}_E \xrightarrow{\Theta} U_A \times_{U_B} \mathfrak{u}_E \xrightarrow{\alpha^E} T(U_A/U_B)$$

is a U_A-equivariant diffeomorphism between the complexification G_A/G_B and the tangent bundle $T(U_A/U_B)$ of the homogeneous space U_A/U_B. Under the above identification the involution σ_G is the map $V \mapsto -V$, $T(U_A/U_B) \to T(U_A/U_B)$.

Remark 2.3.16. *The isomorphism in Remark 2.3.15 gives the tangent bundle of U_A/U_B a complex manifold structure which depends on the map E. With this complex manifold structure the map $T(U_A/U_B) \to T(U_A/U_B)$, $V \mapsto -V$ is anti-holomorphic as in the case of Lempert* ***adapted complex structures*** *which where first studied in [39]. If M is an analytic Riemannian manifold then a complex structure on a disc bundle*

$$T^R M = \{V \in TM : \|V\| < R\}$$

for an $R > 0$ is called adapted if for every unit speed geodesic $\gamma : I \to M$ the map

$$\phi_\gamma : a + bi \mapsto \gamma_{*a}(b\frac{d}{dt}) \quad \text{for } a \in I \text{ and } b \in (-R,R)$$

is holomorphic. In this case the complex structure is unique and the map $T^R M \to T^R M$, $V \mapsto -V$ is antiholomorphic. The complex structure of Remark 2.3.15 is global but not canonical.

The following proposition shows that the diffeomorphism between G_A/G_B and $T(U_A/U_B)$ respects the natural morphisms that can be defined between homogeneous spaces of the form G_A/G_B and tangent bundles of homogeneous spaces given by $T(U_A/U_B)$.

Proposition 2.3.17. *Let $(G_A, G_B; E; \sigma)$ and $(\tilde{G_A}, \tilde{G_B}; \tilde{E}; \tilde{\sigma})$ be reductive structures with involution that satisfy the conditions of the previous remark and let $\alpha : G_A \to \tilde{G_A}$ be a holomorphic morphism of reductive structures with involution. If we define α_G :*

$G_A/G_B \to \tilde{G_A}/\tilde{G_B}$, $gG_B \mapsto \alpha(g)\tilde{G_B}$ and $\alpha_U : U_A/U_B \to \tilde{U_A}/\tilde{U_B}$, $uU_B \mapsto \alpha(u)\tilde{U_B}$ then the diagram

$$\begin{array}{ccccc} G_A/G_B & \xleftarrow{\sim} & U_A \times_{U_B} \mathfrak{u}_E & \xrightarrow{\sim} & T(U_A/U_B) \\ \downarrow{\scriptstyle \alpha_G} & & & & \downarrow{\scriptstyle \alpha_{U*}} \\ \tilde{G_A}/\tilde{G_B} & \xleftarrow{\sim} & \tilde{U_A} \times_{\tilde{U_B}} \tilde{\mathfrak{u}_E} & \xrightarrow{\sim} & T(\tilde{U_A}/\tilde{U_B}) \end{array}$$

commutes, where the horizontal arrows correspond to the morphisms of Rem. 2.3.15.

Proof. Since $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$, $\alpha(U_B) \subseteq \tilde{U_B}$ and α_U is well defined. Since $\alpha_{*1} \circ \sigma_{*1} = \tilde{\sigma}_{*1} \circ \alpha_{*1}$, $\alpha_{*1}(\mathfrak{u}) \subseteq \tilde{\mathfrak{u}}$. Also $E \circ \alpha_{*1} = \alpha_{*1} \circ E$ implies $\alpha_{*1}(KerE) \subseteq Ker\tilde{E}$ so that $\alpha_{*1}(\mathfrak{u}_E) \subseteq \tilde{\mathfrak{u}_E}$. Given $u \in U_A$ and $X \in \mathfrak{u}_E$, $\alpha(u) \in \tilde{U_A}$ and $\alpha_{*1}X \in \tilde{u_E}$ and we have the following diagram

$$\begin{array}{ccccc} ue^{iX}G_B & \longleftarrow\!\shortmid & [(u,X)] & \longmapsto & (\mu_u)_{*o}q_{*1}X \\ \downarrow{\scriptstyle \alpha_G} & & & & \downarrow{\scriptstyle \alpha_{U*}} \\ \alpha(u)e^{i\alpha_{*1}(X)}\tilde{G_B} & \longleftarrow\!\shortmid & [(\alpha(u), \alpha_{*1}(X))] & \longmapsto & (\tilde{\mu}_{\alpha(u)})_{*o}\tilde{q}_{*1}\alpha_{*1}X. \end{array}$$

It is enough to verify that the values in the vertical arrows correspond to the stated morphisms. Note that

$$\alpha_G(ue^{iX}G_B) = \alpha(u)e^{\alpha_{*1}(iX)}\tilde{G_B} = \alpha(u)e^{i\alpha_{*1}(X)}\tilde{G_B}$$

since $\alpha_{*1}(iX) = i\alpha_{*1}(X)$ because α is holomorphic. Since $\alpha_U \circ \mu_u = \tilde{\mu}_{\alpha(u)} \circ \alpha_U$ and $\tilde{q} \circ \alpha = \alpha_U \circ q$ we get

$$\alpha_{U*q(u)}(\mu_u)_{*o}q_{*1}X = (\tilde{\mu}_{\alpha(u)})_{*o}\alpha_{U*o}q_{*1}X = (\tilde{\mu}_{\alpha(u)})_{*o}\tilde{q}_{*1}\alpha_{*1}X.$$

□

Remark 2.3.18. *Observe that the construction of maps in Proposition 2.3.17 are functorial. If $(G_A, G_B; E; \sigma)$, $(\tilde{G_A}, \tilde{G_B}; \tilde{E}; \tilde{\sigma})$ and $(\hat{G_A}, \hat{G_B}; \hat{E}; \hat{\sigma})$ are reductive structures with involution, and $\alpha : G_A \to \tilde{G_A}$ and $\beta : \tilde{G_A} \to \hat{G_A}$ are morphisms of reductive structures with involution we can define α_G, β_G, $(\beta \circ \alpha)_G$, α_U, β_U and $(\beta \circ \alpha)_U$ in the same way as in Proposition 2.3.17. Then*

$$\beta_G \circ \alpha_G = (\beta \circ \alpha)_G \text{ and } \beta_{U*} \circ \alpha_{U*} = (\beta_U \circ \alpha_U)_* = (\beta \circ \alpha)_{U*}.$$

Also $(id_{G_A})_G = id_{G_A/G_B}$ and $((id_{G_A})_U)_* = (id_{U_A/U_B})_* = id_{T(U_A/U_B)}$.

2.3.3 Examples of homogeneous spaces

There are two basic examples of homogeneous spaces U_A/U_B in the infinite dimensional context, the flag manifolds and the Stiefel manifolds. Coadjoint orbits of classical Banach-Lie groups of compact operators are examples of flag manifolds.

Example 2.3.19. *Flag manifolds*

*Let $\mathcal{H}$ be a Hilbert space and let p_i, $i = 1, \dots, n$ be pairwise orthogonal projections in $\mathcal{B}(\mathcal{H})$ each with range $\mathcal{H}_i$ such that $\sum_{i=1}^n p_i = 1$. If we consider the action of the unitary group U_A of $\mathcal{B}(\mathcal{H})$ on the set of n-tuples of pairwise orthogonal projections with sum 1 given by $u \cdot (q_1, \dots, q_n) = (uq_1u^*, \dots, uq_nu^*)$ then the orbit of $(p_1, \dots, p_n)$ can be considered as an infinite dimensional version of a **flag manifold**. This orbit is isomorphic to U_A/U_B where*

$$U_B = \left\{ \begin{pmatrix} u_1 & 0 & \dots & 0 \\ 0 & u_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_n \end{pmatrix} : u_i \text{ unitary in } \mathcal{B}(\mathcal{H}_i) \text{ for } i = 1, \dots, n \right\};$$

and we write the operators in $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n)$ as $n \times n$-matrices with corresponding operator entries. If we consider the group G_A of invertible operators in $\mathcal{B}(\mathcal{H})$ with the usual involution σ, the involutive subgroup

$$G_B = \left\{ \begin{pmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_n \end{pmatrix} : g_i \text{ invertible in } \mathcal{B}(\mathcal{H}_i) \text{ for } i = 1, \dots, n \right\};$$

and the conditional expectation $E : \mathfrak{g}_A \to \mathfrak{g}_B$, $X \mapsto \sum_{i=1}^n p_i X p_i$ then we are in the context of Example 2.2.9 and Theorem 2.3.9. Therefore Theorem 2.3.12 and Remark 2.3.15 give a geometric description of the complexification of the flag manifold.

In Section 6 and 7 of [6] the reader can find further examples of generalized flag manifolds and in [27, 28] the metric geometry of some generalized Grassmann manifolds is studied.

Remark 2.3.20. *The case of the flag manifold with two projections is the infinite dimensional **Grassmannian**. The case of the Grassmannian where the decomposition of $\mathcal{H}$ is $\mathcal{H} = \mathbb{C}\eta \oplus (\mathbb{C}\eta)^\perp$ for a non-zero vector $\eta \in \mathcal{H}$ is the **projective space** $\mathbb{P}(\mathcal{H})$.*

Grassmannians have the structure of a symmetric space. If $\mathcal{H}$ is a Hilbert space then we can identify the set of subspaces of $\mathcal{H}$ with the set of selfadjoint involutions in $\mathcal{B}(\mathcal{H})$ which we denote by $\mathcal{I}$. A subspace $\mathcal{K} \subseteq \mathcal{H}$ corresponds to the selfadjoint involution

$$e_{\mathcal{K}} = 2p_{\mathcal{K}} - 1 = p_{\mathcal{K}} - p_{\mathcal{K}^\perp} : \mathcal{K} \oplus \mathcal{K}^\perp \to \mathcal{K} \oplus \mathcal{K}^\perp, \quad (\xi, \eta) \mapsto (\xi, -\eta)$$

where $p_{\mathcal{K}}$ is the orthogonal projection onto $\mathcal{K}$. The manifold $\mathcal{I}$ is a symmetric space if we define a product by

$$e \cdot f = ef^{-1}e = efe$$

for $e, f \in \mathcal{I}$, so that

$$e_{\mathcal{H}_1} \cdot e_{\mathcal{H}_2} = e_{\mathcal{H}_1} e_{\mathcal{H}_2} e_{\mathcal{H}_1} = e_{e_{\mathcal{H}_1}\mathcal{H}_2}$$

for subspaces $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}$.

Other examples of flag manifolds in the infinite dimensional context are coadjoint orbits in operator ideals, which now can be described geometrically.

Example 2.3.21. *Coadjoint orbits*

In the setting of Example 2.2.10 let $1 < p, q < \infty$ such that $1/p + 1/q = 1$. The Lie algebra of the Banach-Lie group $G_{A,p}$ is $\mathfrak{g}_{A,p} = A_p$, the ideal of p-Schatten operators. The Lie algebra of the real Banach-Lie group $U_{A,p}$ is $\mathfrak{u}_{A,p}$, the skew-adjoint p-Schatten operators. The trace provides strong duality pairings $\mathfrak{g}^*_{A,p} \simeq \mathfrak{g}_{A,q}$ and $\mathfrak{u}^*_{A,p} \simeq \mathfrak{u}_{A,q}$.

We denote by $Ad^* : G_{A,p} \mapsto \mathcal{B}(\mathfrak{g}_{A,q})$, $Ad^*_g(X) = (Ad_{g^{-1}})^*(X) = gXg^{-1}$ for $g \in G_{A,p}$ and $X \in \mathfrak{g}^*_{A,p} \simeq \mathfrak{g}_{A,q}$, the coadjoint action of $G_{A,p}$ and by $Ad^* : U_{A,p} \mapsto \mathcal{B}(\mathfrak{u}_{A,q})$, $Ad^*_u(X) = (Ad_{u^{-1}})^*(X) = uXu^{-1}$ for $u \in U_{A,p}$ and $X \in \mathfrak{u}^*_{A,p} \simeq \mathfrak{u}_{A,q}$, the coadjoint action of $U_{A,p}$.

For a fixed $X \in \mathfrak{u}_{A,q} \subseteq \mathfrak{g}_{A,q}$ let $\mathcal{O}_G(X) = \{Ad^*_g(X) : g \in G_{A,p}\}$ be the ***coadjoint orbit*** *of X under the action of $G_{A,p}$ and $\mathcal{O}_U(X) = \{Ad^*_u(X) : g \in U_{A,p}\}$ be the coadjoint orbit of X under the action of $U_{A,p}$. Since X is a compact skew-adjoint operator it is diagonalizable, i.e. there is a finite or countable sequence of pairwise orthogonal projections $(p_i)_{i=1}^N$ with $N \in \mathbb{N} \cup \{\infty\}$ such that $\sum_{i=1}^N p_i = 1$ and $X = \sum_{i=1}^N \lambda_i p_i$, where $\lambda_i \neq \lambda_j$ for $i \neq j$ and $(\lambda_i)_{i=1}^N \subseteq i\mathbb{R}$. The map $E : Y \mapsto \sum_{i=1}^N p_i Y p_i$ is a conditional expectation from A onto the C^*-subalgebra $B = \{Y \in A : p_i Y = Y p_i \text{ for all } i \geq 1\}$. This conditional expectation sends trace-class operators to trace-class operators and preserves the trace, so the conditions on E in Example 2.2.10 are satisfied. The coadjoint isotropy group of X for the action of $G_{A,p}$ is $\{g \in G_{A,p} : gXg^{-1} = X\} = G_{B,p}$ and the coadjoint isotropy group of X for the action of $U_{A,p}$ is $\{u \in U_{A,p} : uXu^{-1} = X\} = U_{B,p}$ (this follows from the fact that an operator commutes with a diagonalizable operator if and only if it leaves all the eigenspaces of the diagonalizable operator invariant). Thus, making the identifications $\mathcal{O}_G(X) \simeq G_{A,p}/G_{B,p}$ and $\mathcal{O}_U(X) \simeq U_{A,p}/U_{B,p}$, Theorem 2.3.9, Theorem 2.3.12 and Remark 2.3.15 give a geometric description of the complexification of the flag manifold; there* *is a $U_{A,p}$-equivariant fiber bundle isomorphism between $\mathcal{O}_G(X)$ and $T(\mathcal{O}_U(X))$ covering the identity map of $\mathcal{O}_U(X)$.*

For the case of trace class operators $\mathfrak{g}_{A,1} = A_1$ we have to restrict the coadjoint orbits under consideration to the orbits of compact skew-adjoint operators, since $\mathfrak{g}^*_{A,1} = \mathcal{B}(\mathcal{H})$ an arbirtrary bounded skew-adjoint operators are not diagonalizable.

Likewise, it is now possible to give a geometric description of the complexification of the Stiefel manifolds, see [14] for further information about the metric geometry of Stiefel manifolds.

Example 2.3.22. *Stiefel manifolds*

Let $\mathcal{H}$ be a Hilbert space and let p_i, $i = 1, 2$ be pairwise orthogonal projections in $\mathcal{B}(\mathcal{H})$ each with range $\mathcal{H}_i$ such that $p_1 + p_2 = 1$. If we consider the action of the unitary group U_A of $\mathcal{B}(\mathcal{H})$ on the set of partial isometries given by by $u \cdot v = uv$ then the orbit of p_1 can be considered as an infinite dimensional version of a ***Stiefel manifold****. This orbit is isomorphic to U_A/U_B where*

$$U_B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} : u \text{ is unitary in } \mathcal{B}(\mathcal{H}_2) \right\},$$

and we write the operators in $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ as 2×2-matrices with corresponding operator entries. If we consider the group G_A of invertible operators in $\mathcal{B}(\mathcal{H})$ with the usual involution σ, the involutive subgroup

$$G_B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} : g \text{ is invertible in } \mathcal{B}(\mathcal{H}_2) \right\},$$

and the map $E : \mathfrak{g}_A \to \mathfrak{g}_B$, $X \mapsto (1-p)X(1-p)$ then we are in the context of Example 2.2.11 and Theorem 2.3.9. Therefore Theorem 2.3.12 and Remark 2.3.15 give a geometric description of the complexification of the Stiefel manifold.

Remark 2.3.23. *In the case of the Stiefel manifold where the decomposition of $\mathcal{H}$ is $\mathcal{H} = \mathbb{C}\eta \oplus (\mathbb{C}\eta)^\perp$ for a non-zero vector $\eta \in \mathcal{H}$ we see that $U_A/U_B \simeq \{\xi \in \mathcal{H} : \|\xi\| = 1\}$, the unit sphere in the Hilbert space $\mathcal{H}$.*

The unit sphere in the Hilbert space has the structure of symmetric space with product defined by

$$\xi \cdot \eta = 2\langle \xi, \eta \rangle \xi - \eta$$

for $\xi, \eta \in S$.

Coadjoint orbits in ideal of p-Schatten operators can be endowed with the structure of symplectic manifold. The following is Theorem 7.3 and 7.4 of [49] where Odzijewicz and Ratiu endow coadjoint orbits with symplectic forms. For further reading on the coadjoint orbits in the infinite dimensional setting, see Section 7 in [49] and Section 4 in [8].

Theorem 2.3.24. *Let G be a (real or complex) Banach Lie group with Lie algebra $\mathfrak{g}$. Assume that:*

1. *$\mathfrak{g}$ admits a predual $\mathfrak{g}_*$.*
2. *the coadjoint action of G on the dual $\mathfrak{g}^*$ leaves the predual $\mathfrak{g}_*$ invariant, that is, $Ad_g^*(\mathfrak{g}_*) \subseteq \mathfrak{g}_*$ for any $g \in G$.*
3. *for a fixed $\rho \in \mathfrak{g}_*$ the coadjoint isotropy subgroup $G_\rho = \{g \in G : Ad_g^* \rho = \rho\}$ is a Lie subgroup of G in the sense that it is a submanifold of G.*

Then the Lie algebra of G_ρ equals $\mathfrak{g}_\rho = \{\xi \in \mathfrak{g} : ad_\xi^* \rho = 0\}$ and the quotient topological space G/G_ρ admits a unique (real or complex) Banach manifold structure making the canonical projection $\pi : G \to G/G_\rho$ a surjective submersion. The manifold G/G_ρ is symplectic relative to the 2-form ω_ρ given by

$$\omega_\rho(\pi(g))((\pi \circ L_g)_{*1}\xi, (\pi \circ L_g)_{*1}\eta) = \langle \rho, [\xi, \eta] \rangle$$

for $\xi, \eta \in \mathfrak{g}$ where $\langle \cdot, \cdot \rangle : \mathfrak{g}_* \times \mathfrak{g} \to \mathbb{R}$ or $\mathbb{C}$ is the canonical pairing between $\mathfrak{g}_*$ and $\mathfrak{g}$. The two form ω_ρ is invariant under the action of G on G/G_ρ given by $g \cdot \pi(h) = \pi(gh)$ for $g, h \in G$.

The results in this chapter have been published in [42].

Chapter 3

A geometric approach to similarity problems

3.1 Introduction

In this chapter we study similarity problems geometrically by analyzing the action of a group H of invertible operators in a C^*-algebra A on the positive invertible operators P given by $h \cdot a = hah^*$.

In Section 3.2 we prove basic properties of the action of a group H of invertible elements of a C^*-algebra on the cone of positive invertible elements given by $h \cdot a = hah^*$ and its relation to unitarizers of groups, i.e. positive invertibles s such that $s^{-1}Hs$ is a group of unitaries.

In Section 3.3 we define the similarity number and size of a group and relate it to geometric properties of the orbits of the natural action on P. This geometric approach is used to prove some interpolation results in Pisier's study of similarity problems, so given a unital homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$, a family of unital homomorphisms with norm tending to 1 is derived and the norms and completely bounded norms of these homomorphisms are related to the orbits of the natural action of $\pi(U)$. We also give an answer to a problem posed by Andruchow, Corach and Stojanoff in [2, 4] about the minimality properties of the canonical unitarizers of some representations $g\pi(\cdot)g^{-1}$ where g is an invertible operator and $\pi : A \to \mathcal{B}(\mathcal{H})$ is a $*$-representation such that there is a conditional expectation $E : \mathcal{B}(\mathcal{H}) \to \pi(A)$.

In Section 3.4 we address the question of the unitarizability of uniformly bounded groups in $\mathcal{B}(\mathcal{H})$ in two contexts where the metric on the manifolds of positive invertible operators are derived from a trace: the trace in a finite von Neumann algebra and the trace in the ideal of Hilbert-Schmidt operators. In these contexts of CAT(0) spaces the

Bruhat-Tits fixed point theorem is used to prove similarity results.

3.2 Fixed point set and orbits

Definition 3.2.1. *If A is a C^*-algebra, P is the set of positive invertible elements and G the group of invertible elements of A, then for a subgroup $H \subseteq G$ we define the action I of H on P as $h \cdot a = I_h(a) = hah^*$. To make clear which subgroup H of G acts on P we shall sometimes write I_H. The fixed point set for this action is $P^H = \{a \in P : I_h(a) = a \text{ for all } h \in H\}$. The orbit of $a \in P$ is $\mathcal{O}_H(a) = \{h \cdot a : h \in H\}$. A group H is said to be* ***unitarizable*** *if there is an invertible operator s such that $s^{-1}Hs$ is a group of unitaries.*

Remark 3.2.2. *Note that if s is a unitarizer of H and $s = bu$ is the polar decomposition of s into a product of a positive invertible b and a unitary u, then b is a positive unitarizer of H because $u^{-1}b^{-1}Hbu$ is a group of unitaries. In this case $\|s\| = \|b\|$.*

The next proposition shows how positive unitarizers are related to fixed points of the action I.

Proposition 3.2.3. *A positive invertible operator s is a positive unitarizer of the group H if and only if s^2 is a fixed point for the action I of H on P.*

Proof. Observe that

$$\begin{aligned} s^{-1}Hs \subseteq U &\Leftrightarrow s^{-1}hs(s^{-1}hs)^* = 1 \text{ for all } h \in H \\ &\Leftrightarrow s^{-1}hs^2h^*s^{-1} = 1 \text{ for all } h \in H \\ &\Leftrightarrow I_h(s^2) = hs^2h^* = s^2 \text{ for all } h \in H. \end{aligned}$$

□

We next show how orbits and fixed point sets behave under translations.

Proposition 3.2.4. *Let a group G act on a set X. If H is a subgroup of G then for $f \in G$ and $x \in X$*

$$f^{-1} \cdot \mathcal{O}_H(x) = \mathcal{O}_{f^{-1}Hf}(f^{-1} \cdot x)$$

and

$$f^{-1} \cdot X^H = X^{f^{-1}Hf}.$$

Proof. To prove the first identity observe that

$$\begin{aligned}\mathcal{O}_{f^{-1}Hf}(x) &= \{(f^{-1}hf)\cdot x : h \in H\} \\ &= \{f^{-1}\cdot(h\cdot(f\cdot x)) : h\in H\} \\ &= f^{-1}\cdot\{h\cdot(f\cdot x) : h \in H\} \\ &= f^{-1}\cdot\mathcal{O}_H(f\cdot x).\end{aligned}$$

Substituing $f^{-1}\cdot x$ for x we get the result. The second identity follows from

$$\begin{aligned}x\in X^{f^{-1}Hf} &\Leftrightarrow f^{-1}hf\cdot x = x \text{ for all } h\in H \\ &\Leftrightarrow f^{-1}\cdot(h\cdot(f\cdot x)) = x \text{ for all } h \in H \\ &\Leftrightarrow h\cdot f\cdot x = f\cdot x \text{ for all } h\in H \\ &\Leftrightarrow f\cdot x\in X^H \Leftrightarrow x\in f^{-1}\cdot X^H.\end{aligned}$$

□

Remark 3.2.5. *If A is a C^*-algebra, P is the set of positive invertible elements and G the group of invertible elements of A, then Proposition 3.2.4 says that for a subroup H of G and for $f\in G$ and $a\in P$*

$$I_{f^{-1}}(\mathcal{O}_H(a)) = f^{-1}\mathcal{O}_H(a)f^{-1*} = \mathcal{O}_{f^{-1}Hf}(f^{-1}af^{-1*})$$

and

$$I_{f^{-1}}(p^H) = f^{-1}p^H f^{-1*} = P^{f^{-1}Hf}.$$

Remark 3.2.6. *If H is a group of unitaries in $\mathcal{B}(\mathcal{H})$, then the commutant H' of H in $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra so that*

$$\begin{aligned}P^H &= \{a\in P : I_h(a) = hah^{-1} = a \textit{ for all } h\in H\} \\ &= \{a \in P : ha = ah \textit{ for all } h\in H\} \\ &= P\cap H' = exp(H'\cap A_s).\end{aligned}$$

Definition 3.2.7. *A closed real subspace $S\subseteq A_s \simeq T_1P$ is called a* ***Lie triple system*** *if $[[X,Y],Z]\in S$ for every $X,Y,Z\in S$. A closed submanifold $C\subseteq P$ is* ***totally geodesic*** *if $exp_a(T_aC) = C$ for all $a\in C$.*

Proposition 3.2.8. *Let H be a group of invertible elements, then the fixed point set P^H of the action I is a totally geodesic submanifold of P.*

Proof. If H is not unitarizable then P^H is empty. If H is unitarizable and f is a positive unitarizer, then by Prop. 3.2.5 $P^H = fP^{f^{-1}Hf}f$, so that the fixed point set is a translation of the fixed point set of the unitary group $f^{-1}Hf$. By Remark 3.2.6 $P^{f^{-1}Hf} = P \cap (f^{-1}Hf)' = exp((f^{-1}Hf)' \cap A_s)$. Since $(f^{-1}Hf)'$ is a $*$-subalgebra of A it is a Lie triple system. From the identity $[X,Y]^* = -[X^*, Y^*]$ it is easy to verify that A_s is e Lie triple system. Therefore the intersection $(f^{-1}Hf)' \cap A_s$ is a Lie triple system and by Corollary 4.17 in [15] $P^{f^{-1}Hf} = P \cap (f^{-1}Hf)' = exp((f^{-1}Hf)' \cap A_s)$, beeing the exponential of a Lie triple system, is a totally geodesic submanifold. Since P^H is a translation of the totally geodesic subset $P^{f^{-1}Hf}$ it is also totally geodesic. □

3.3 Similarity number and size of a group

3.3.1 Geometric characterization of the similarity number and size of a group

Recall that by Proposition 1.5.9 the action I of G on P given by $g \cdot a = gag^*$ is isometric and that by Proposition 1.6.11 the distance between two positive invertibles elements a and b is given by

$$d(a,b) = Length(\gamma_{a,b}) = \|log(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})\|.$$

Definition 3.3.1. *For subsets $C, D \subseteq P$ and $a \in P$ we define as usual $dist(C,D) = inf_{x\in C, y\in D} d(x,y)$, $dist(a,D) = inf_{x\in D} d(a,x)$ and $diam(D) = sup_{x,y\in D} d(x,y)$.*

Definition 3.3.2. *The* ***size of a group*** *$H \subseteq G$ is $|H| = sup_{h\in H}\|h\|$. The* ***similarity number*** *of H is $Sim(H) = \inf\{\|s\|\|s^{-1}\| : s \text{ is a unitarizer of } H\}$.*

The similarity number defined above is not the same as the similarity degree defined and used in Pisier's approach to similarity problems in [56, 55]. By Remark 3.2.2 it is straightforward to check that

$$Sim(H) = \inf\{\|s\|\|s^{-1}\| : s \text{ is a positive unitarizer of } H\}$$

.

Proposition 3.3.3. *For a group H the identities*

$$dist(\mathcal{O}_H(1), P^H) = dist(1, P^H) = log(Sim(H))$$

and

$$diam(\mathcal{O}_H(1)) = 2log(|H|)$$

hold.

Proof. We donote by $\lambda_{max}(a)$ and by $\lambda_{min}(a)$ the maximum and the minimum of the spectrum of $a \in P$. Then, using the characterization of unitarizers

$$\begin{aligned} Sim(H) &= \inf\{\|s\|\|s^{-1}\| : s \text{ is a positive unitarizer of } H\} \\ &= \inf_{a \in P^H} \|a^{\frac{1}{2}}\|\|a^{-\frac{1}{2}}\| \text{ by Proposition 3.2.3} \qquad (3.1) \\ &= \inf_{a \in P^H} \left(\frac{\lambda_{max}(a)}{\lambda_{min}(a)}\right)^{\frac{1}{2}}. \end{aligned}$$

Also, using the fact that for $a \in P^H$ and $\alpha > 0$ we have $\alpha a \in P^H$

$$\begin{aligned} dist(1, P^H) &= \inf_{a \in P^H} d(1, a) = \inf_{a \in P^H} \|log(a)\| \\ &= \inf_{a \in P^H} max\{log(\lambda_{max}(a)), -log(\lambda_{min}(a))\} \\ &= \inf_{a \in P^H, \alpha > 0} max\{log(\lambda_{max}(\alpha a)), -log(\lambda_{min}(\alpha a))\} \qquad (3.2) \\ &= \inf_{a \in P^H, \alpha > 0} max\{log(\lambda_{max}(a)) + log(\alpha), -log(\lambda_{min}(a)) - log(\alpha)\} \\ &= \inf_{a \in P^H, c \in \mathbb{R}} max\{log(\lambda_{max}(a)) + c, -log(\lambda_{min}(a)) - c\} \\ &= \inf_{a \in P^H} \frac{1}{2}(log(\lambda_{max}(a)) - log(\lambda_{min}(a))) \\ &= log\left(\inf_{a \in P^H} (\frac{\lambda_{max}(a)}{\lambda_{min}(a)})^{\frac{1}{2}}\right). \end{aligned}$$

Combining (3.1) and (3.2) we get

$$dist(1, P^H) = log(Sim(H)).$$

Also

$$\begin{aligned} dist(\mathcal{O}_H(1), P^H) &= inf_{h \in H} dist(I_h(1), P^H) \\ &= inf_{h \in H} dist(1, I_{h^{-1}}(P^H)) \\ &= inf_{h \in H} dist(1, P^H) \\ &= dist(1, P^H), \end{aligned}$$

where the second equality follows from the fact that I is isometric, and the third equality follows from the fact that P^H is I invariant.

Since

$$\begin{aligned} d(1, hh^*) &= \|log(hh^*)\| = max\{log\|hh^*\|, log\|(hh^*)^{-1}\|\} \\ &= max\{log(\|h\|^2), log(\|h^{-1}\|^2)\|\} \end{aligned}$$

we get

$$\begin{aligned} diam(\mathcal{O}_H(1)) &= \sup_{h\in H} d(1, hh^*) \\ &= \sup_{h\in H} max\{log(\|h\|^2), log(\|h^{-1}\|^2)\|\} \\ &= \sup_{h\in H} log(\|h\|^2) \\ &= \sup_{h\in H} 2log(\|h\|) \\ &= 2log(|H|). \end{aligned}$$

□

Remark 3.3.4. *Note from the proof of $dist(1, P^H) = log(Sim(H))$ that an $a \in P^H$ which minimizes the distance to 1 corresponds to a unitarizer $a^{\frac{1}{2}}$ which minimizes the quantity $\|s\|\|s^{-1}\|$ among all unitarizers. Also, a unitarizer s such that $\|s\|\|s^{-1}\| = Sim(H)$ can be scaled to have simetric spectrum, i.e. $log(\lambda_{max}(s)) = -log(\lambda_{min}(s))$ and the resulting scaled fixed point s^2 minimizes the distance to 1.*

Proposition 3.3.3 was proved independently by Schlicht (see Lemma 5.2 and the proof of Lemma 5.6 in [59]). The next lemma proved by Schlicht in the case of $\mathcal{B}(\mathcal{H})$ (see Lemma 3.4 in [59]) shows that closest points to the identity 1 in P^H exist. We include a proof in the case of von Neumann algebras. Note that this is equivalent by Proposition 3.3.3 to proving that for a unitarizable group H there is a positive unitarizer s such that $\|s\|\|s^{-1}\| = Sim(H)$.

Lemma 3.3.5. *Let A be a von Neumann algebra with separable predual and let H be a subgroup of G, then there is an $a \in P^H$ such that $dist(1, P^H) = d(1, a)$.*

Proof. For $a \in P$

$$d(1, a) = \|log(a)\| = max\{log(\lambda_{max}(a)), -log(\lambda_{min}(a))\},$$

where $\lambda_{max}(a)$ and $\lambda_{min}(a)$ denote the maximum and minimum eigenvalues of $a \in P \subseteq A_s$. Hence the metric balls around 1 are operator intervals, i.e.

$$B[1, r] = \{b \in P : d(1, b) \leq r\} = [e^{-r}, e^r].$$

There is a sequence $(a_n)_n \subseteq P^H$ such that $d(1, a_n) \to dist(1, P^H) = \inf_{b \in P^H} d(1, b)$. Since the set

$$\{a \in A : hah^* = a \text{ for all } h \in H\} = \bigcap_{h \in H} \{a \in A : hah^* = a\}$$

is weak operator closed, and for every $r > 0$ the set $[e^{-r}, e^r]$ is is also weak operator closed we conclude that $P^H \cap [e^{-r}, e^r]$ is weak operator closed. Also, since the weak operator topology on closed balls is metrizable and compact it follows that there is a subsequence of $(a_n)_n$ which converges weakly to an $a \in P^H$. This subsequence, which we still denote by $(a_n)_n$, also satisfies $d(1, a_n) \to dist(1, P^H) = \inf_{b \in P^H} d(1, b)$. For every $\epsilon > 0$ there is an $n_\epsilon \in \mathbb{N}$ such that for $n \geq n_\epsilon$ we have

$$a_n \in B[1, dist(1, P^H) + \epsilon] = [e^{-dist(1,P^H)-\epsilon}, e^{dist(1,P^H)+\epsilon}].$$

Since operator intervals are weak operator closed it follows that the weak limit a of $(a_n)_n$ is in $[e^{-dist(1,P^H)-\epsilon}, e^{dist(1,P^H)+\epsilon}]$. Therefore $d(1, a) < dist(1, P^H) + \epsilon$ for every $\epsilon > 0$ so that $d(1, a) \leq dist(1, P^H)$. Since $d(1, a) \geq dist(1, P^H) = \inf_{b \in P^H} d(1, b)$ the conclusion follows. □

3.3.2 Geometric interpretation of similarity results

Similarity results for homomorphisms of C^*-algebras can be obtained by restricting attention to the group of unitaries in the C^*-algebra. The following is Lemma 9.6 in [51].

Proposition 3.3.6. *If A and B are unital C^*-algebras and $\pi : A \to B$ is a unital homomorphism, then π is a $*$-homomorphism if and only if π sends unitaries to unitaries, i.e. $\pi(U_A) \subseteq U_B$, where U_A and U_B are the group of unitaries of A and B respectively.*

Note that homomorphisms are algebra homomorphism which not necessarily preserve the $*$-operation.

Proof. If π sends unitaries to unitaries then for every $u \in U_A$

$$\pi(u^*) = \pi(u^{-1}) = \pi(u)^{-1} = \pi(u)^*$$

so that π preserves the $*$ operator on unitaries. Since every element of A is a real linear combination of four unitaries (see Proposition 13.3 in [17]) and π is linear we conclude that π is a $*$-homomorphism. The other implication is clear. □

Definition 3.3.7. *For a C^*-algebra A and an inverible $s \in A$ we define the unital bounded homomorphism*

$$Ad_s : A \mapsto A, \quad Ad_s(a) = sas^{-1} \text{ for } a \in A.$$

Corollary 3.3.8. *If A and B are unital C^*-algebras and $\pi : A \to B$ is a unital homomorphism, then $Ad_s \circ \pi$ is a $*$-homomorphism for an invertible $s \in B$ if and only if $Ad_s(\pi(U_A)) = s^{-1}\pi(U_A)s$ is a group of unitaries.*

Proposition 3.3.9. *If A and B are unital C^*-algebras and $\pi : A \to B$ is a unital homomorphism, then $|\pi(U_A)| = \|\pi\|$.*

Proof. That $|\pi(U_A)| \leq \|\pi\|$ is clear. To prove that $\|\pi\| \leq |\pi(U_A)|$ we use the fact that in a C^*-algebra the closed unit ball is the closed convex hull of unitaries in the algebra, see [24, Theorem I.8.4]. If $a \in A$ is such that $\|a\| \leq 1$, for $\epsilon > 0$ there is a convex combination of unitaries $\sum_{i=1}^n \alpha_i u_i$ in A such that $\|a - \sum_{i=1}^n \alpha_i u_i\| \leq \epsilon$. Hence $\|\pi(a)\| - \|\pi(\sum_{i=1}^n \alpha_i u_i)\| \leq \|\pi(a) - \pi(\sum_{i=1}^n \alpha_i u_i)\| \leq \|\pi\| \|a - \sum_{i=1}^n \alpha_i u_i\| \leq \|\pi\|\epsilon$ and $\|\pi(a)\| \leq \|\pi(\sum_{i=1}^n \alpha_i u_i)\| + \epsilon\|\pi\|$. Since $\|\pi(\sum_{i=1}^n \alpha_i u_i)\| = \|\sum_{i=1}^n \alpha_i \pi(u_i)\| \leq \sum_{i=1}^n \alpha_i \|\pi(u_i)\| \leq \sum_{i=1}^n \alpha_i |\pi(U_A)| = |\pi(U_A)|$ the conclusion follows. □

If a C^*-algebra A is represented by means of a one-to-one $*$-homomorphism $\psi : A \to \mathcal{B}(\mathcal{H})$, then for $n \in \mathbb{N}$ an $n \times n$ operator matrix $(\psi(a_{i,j}))_{i,j=1}^n$ acts naturally on $\mathcal{H}^{(n)} = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$ (n times) and has therofore a C^*-algebra norm inherited from $\mathcal{B}(\mathcal{H}^{(n)})$. It is easy to check that this C^*-algebra of $n \times n$ operator matrices does not depend on the particular choice of representation ψ and we denote it by $M_n(A)$, or using tensor notation $A \otimes M_n(\mathbb{C})$.

Definition 3.3.10. *If A and B are two unital C^*-algebras and $\pi : A \to B$ is a linear map, the **completely bounded norm** of π is $\|\pi\|_{c.b.} = sup_{n\in\mathbb{N}}\|\pi_n\|$, where*

$$\pi_n = \pi \otimes Id_{M_n(\mathbb{C})} : M_n(A) = A \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C}) = M_n(B)$$

$$(a_{i,j})_{i,j=1}^n \mapsto (\pi(a_{i,j}))_{i,j=1}^n.$$

*If $\|\pi\|_{c.b.} < \infty$ then π is a **completely bounded map**.*

The following result is due to Haagerup, see Theorem 1.10 [32].

Theorem 3.3.11. *Let A be a C^*-algebra with unit and let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a bounded unital homomorphism. Then π is similar to a $*$-homomorphism (i.e. there is an invertible $s \in \mathcal{B}(\mathcal{H})$ such that $Ad_s \circ \pi$ is a $*$-homomorphism) if and only if π is completely bounded. If π is completely bounded then*

$$\|\pi\|_{c.b.} = inf\{\|s^{-1}\|\|s\| : Ad_s \circ \pi \text{ is a } *\text{-homomorphism } \}.$$

Proposition 3.3.12. *Let A be a C^*-algebra with unit and let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a completely bounded unital homomorphism. Then*

$$\|\pi\|_{c.b.} = Sim(\pi(U_A)) = exp(dist(1, P^{\pi(U_A)})).$$

Proof.

$$\begin{aligned}
\|\pi\|_{c.b.} &= inf\{\|s\|\|s^{-1}\| : Ad_s \circ \pi \text{ is a } *\text{-homomorphism } \} \text{ by Theorem 3.3.11} \\
&= inf\{\|s\|\|s^{-1}\| : s \text{ is a unitarizer of } \pi(U_A)\} \text{ by Corollary 3.3.8} \\
&= Sim(\pi(U_A)) \text{ by Definition 3.3.2} \\
&= exp(dist(1, P^{\pi(U_A)})) \text{ by Proposition 3.3.3.}
\end{aligned}$$

□

Pisier used bounds that relate the similarity number and size of groups to characterize classes of groups and algebras, see Theorem 1 in [53] and the discussion following that theorem. If we take the logarithm in inequalities of the form

$$Sim(H) \leq K|H|^{\alpha}$$

for positive constants K and α we get by Proposition 3.3.3

$$dist(1, P^H) \leq log(K) + \frac{\alpha}{2} D_H(1).$$

Recall that a C^*-algebra A is ***nuclear*** if for every C^*-algebra B the tensor product algebra $A \otimes B$ has a unique C^*-algebra norm, see Theorem 3.8.7 in [11]. Theorem 1 in [53] becomes

Theorem 3.3.13. *A C^*-algebra A is nuclear if and only if for every unital completely bounded homomorphism $\psi : A \to \mathcal{B}(\mathcal{H})$*

$$dist(1, P^{\psi(U_A)}) \leq D_{\psi(U_A)}(1)$$

where U_A is the group of unitaries of A. A C^*-algebra A is finite dimensional if and only if there is a $c > 0$ such that for every unital completely bounded homomorphism $\psi : A \to \mathcal{B}(\mathcal{H})$

$$dist(1, P^{\psi(U_A)}) \leq c + \frac{1}{2} D_{\psi(U_A)}(1).$$

Remark 3.3.14. *The new statement of Theorem 3.3.13 has therefore a geometric intepretation in terms of metric properties of the orbits of the action $I \circ \psi$ of U_A on P.*

A similar translation of Pisier's characterizations of amenable and finite discrete groups was obtained by Schlicht, see Corollary 5.8 in [59].

3.3.3 Geometric interpolation for the similarity number and size of a group

Instead of the complex interpolation techniques used by Pisier [55, Lemma 2.2 and Lemma 2.3] we use geometric interpolation.

Definition 3.3.15. *For a uniformly bounded group of invertible elements H in a C^*-algebra A let $D_H : P \to \mathbb{R}^+$ be defined by $D_H(a) = diam(\mathcal{O}_H(a))$ for $a \in P$, so that $D_H(a)$ is the diameter of the the orbit that contains a.*

Proposition 3.3.16. *The map $D_H : P \to \mathbb{R}^+$ is invariant for the action of I, geodesically convex and 2-Lipschitz.*

Proof. That D_H is invariant follows from the fact that $\mathcal{O}_H(h \cdot a) = \mathcal{O}_H(a)$ for $a \in P$ and $h \in H$.

To prove that D_H is geodesically convex we see that for a geodesic $\gamma_{a,b} : [0,1] \to P$ the following holds

$$
\begin{aligned}
D_H(\gamma_{a,b}(t)) &= sup_{h\in H} d(\gamma_{a,b}(t), h \cdot \gamma_{a,b}(t)) \\
&= sup_{h\in H} d(\gamma_{a,b}(t), \gamma_{h\cdot a, h\cdot b}(t)) \\
&\leq sup_{h\in H}(td(a, h\cdot a) + (1-t)d(b, h\cdot b)) \\
&\leq t sup_{h\in H} d(a, h\cdot a) + (1-t) sup_{h\in H} d(b, h\cdot b) \\
&= tD_H(a) + (1-t)D_H(b)
\end{aligned}
$$

where the first inequality follows from the fact that the distance along geodesics is convex, see Proposition 1.6.6.

To prove that D_H is 2-Lipschitz observe that

$$
\begin{aligned}
D_H(a) &= sup_{h\in H} d(a, h\cdot a) \\
&\leq sup_{h\in H}(d(a,b) + d(b, h\cdot b) + d(h\cdot b, h\cdot a)) \\
&= sup_{h\in H}(2d(a,b) + d(b, h\cdot b)) \\
&= 2d(a,b) + sup_{h\in H} d(b, h\cdot b) \\
&= 2d(a,b) + D_H(b).
\end{aligned}
$$

Therefore $D_H(a) - D_H(b) \leq 2d(a,b)$. By symmetry $D_H(b) - D_H(a) \leq 2d(b,a)$ so that $|D_H(a) - D_H(b)| \leq 2d(a,b)$. □

Remark 3.3.17. *For a geodesic γ in P the quotient*

$$f_\gamma(t) = \frac{D_H(\gamma(t)) - D_H(\gamma(0))}{d(\gamma(t), \gamma(0))}$$

is a convex function of t because D_H is geodesically convex. It is bounded above by 2 and bounded below by -2 because D_H is 2-Lipschitz. Therefore the limit of $f_\gamma(t)$ when $t \to \infty$ exists and we can interprete this quantity as a slope of D_H at infinity.

Proposition 3.3.18. *For a geodesically convex subset $C \subseteq P$ the map*

$$P \to \mathbb{R}^+, \quad a \mapsto dist(a, C)$$

is geodesically convex and 1-Lipshitz.

Proof. Let $\epsilon > 0$ and let $e, f \in C$ such that $d(a, e) < d(a, C) + \frac{\epsilon}{2}$ and $d(b, f) < d(a, C) + \frac{\epsilon}{2}$. Since $\gamma_{e,f}$ lies in C we have for $t \in [0, 1]$

$$\begin{aligned} dist(\gamma_{a,b}(t), C) &\leq dist(\gamma_{a,b}(t), \gamma_{e,f}(t)) \leq (1-t)d(a, e) + td(b, f) \\ &\leq (1-t)dist(a, C) + tdist(b, C) + \epsilon. \end{aligned}$$

Taking $\epsilon > 0$ arbitrary small we get the inequality. Observe also that

$$d(a, C) \leq inf_{c \in C}(d(a, b) + d(b, c)) = d(a, b) + d(b, C),$$

so that by symmetry we get the Lipschitz bound. □

Theorem 3.3.19. *If H is a uniformly bounded group, $\gamma_t = \gamma_{r^2,s^2}(t)$ is the geodesic connecting positive invertible elemets r^2 and s^2 and $H_t = \gamma_t^{-\frac{1}{2}} H \gamma_t^{\frac{1}{2}}$ is the one-parameter family of groups between the group $r^{-1}Hr$ and the group $s^{-1}Hs$ then*

$$|H_t| \leq |r^{-1}Hr|^{1-t} |s^{-1}Hs|^t$$

If H is a unitarizable group, $\gamma_t = \gamma_{r^2,s^2}(t)$ is the geodesic connecting positive invertible elemets r^2 and s^2 and $H_t = \gamma_t^{-\frac{1}{2}} H \gamma_t^{\frac{1}{2}}$ is the one-parameter family of groups between the group $r^{-1}Hr$ and the group $s^{-1}Hs$ then

$$Sim(H_t) \leq Sim(r^{-1}Hr)^{1-t} Sim(s^{-1}Hs)^t$$

If H is a unitarizable group, and s is a positive unitarizer such that $d(1, P^H) = d(1, s^2)$ (and therefore $\|s\|\|s^{-1}\| = Sim(H)$ by Remark 3.3.4), then the family of groups $(H_t)_{t \in [0,1]}$ with $H_t = s^{-t} H s^t$ satisfies

$$|H_t| \leq |H|^{1-t}$$

and

$$Sim(H_t) = Sim(H)^{1-t}.$$

Proof. By Proposition 3.2.4 for $f \in G$ and $b \in P$

$$\begin{aligned} D_{f^{-1}Hf}(b) &= diam(\mathcal{O}_{f^{-1}Hf}(b)) = diam(f^{-1}\mathcal{O}_H(fbf^*)f^{-1*}) \\ &= diam(\mathcal{O}_H(fbf^*)) = D_H(fbf^*). \end{aligned}$$

Now, using the fact that $\gamma_t = \gamma_{r^2,s^2}(t)$ is a geodesic and the geodesic convexity of D_H

$$\begin{aligned} D_{H_t}(1) &= D_{\gamma_t^{-\frac{1}{2}}H\gamma_t^{\frac{1}{2}}}(1) = D_H(\gamma_t) \\ &= D_H(\gamma_{r^2,s^2}(t)) \leq (1-t)D_H(r^2) + tD_H(s^2) \\ &= (1-t)D_{r^{-1}Hr}(1) + tD_{r^{-1}Hr}(1). \end{aligned}$$

Exponentiating this equation and using Proposition 3.3.3 we get

$$|H_t|^2 \leq |r^{-1}Hr|^{2(1-t)}|s^{-1}Hs|^{2t}$$

and therefore

$$|H_t| \leq |r^{-1}Hr|^{1-t}|s^{-1}Hs|^t.$$

By Proposition 3.2.4 for $f \in G$ and $b \in P$

$$\begin{aligned} D_{f^{-1}Hf}(b) &= diam(\mathcal{O}_{f^{-1}Hf}(b)) = diam(f^{-1}\mathcal{O}_H(fbf^*)f^{-1*}) \\ &= diam(\mathcal{O}_H(fbf^*)) = D_H(fbf^*) \end{aligned}$$

and

$$dist(b, P^{f^{-1}Hf}) = dist(b, f^{-1}p^H f^{-1*}) = dist(fbf^*, P^H).$$

Since P^H is geodesically convex we can use the convexity of the map $a \mapsto d(a,)$, therefore

$$\begin{aligned} dist(1, P^{H_t}) &= dist(1, P^{\gamma_t^{-\frac{1}{2}}H\gamma_t^{\frac{1}{2}}}) \\ &= dist(\gamma_t, P^H) = dist(\gamma_{r^2,s^2}(t), P^H) \\ &\leq (1-t)d(r^2, P^H) + td(s^2, P^H) \\ &= (1-t)dist(1, P^{r^{-1}Hr}) + tdist(1, P^{s^{-1}Hs}). \end{aligned}$$

Exponentiating this inequality we obtain

$$Sim(H_t) \leq Sim(r^{-1}Hr)^{1-t}Sim(s^{-1}Hs)^t.$$

Now, if the geodesic is $\gamma_{1,s^2}(t) = s^{2t}$, then since $H_1 = s^{-1}Hs$ is a group of unitary

$$|H_t| \leq |H|^{1-t}.$$

In the inequality for the similarity number we can get instead an equality. Since s^2 is a point in P^H which minimizes the distance from 1 to P^H and geodesic have minimal lenght, s^2 minimizes distance between the points in P^H to any point in the geodesic $\gamma_{1,s^2}(t) = s^{2t}$. Therefore

$$dist(1, P^{H_t}) = dist(1, P^{\gamma_t^{-\frac{1}{2}} H \gamma_t^{\frac{1}{2}}}) = dist(\gamma_t, P^H) = (1-t)dist(1, P^H).$$

Exponentiating this equation and using Proposition 3.3.3 we get

$$Sim(H_t) = Sim(H)^{1-t}.$$

□

Corollary 3.3.20. *In the case of a completely bounded unital map $\pi : A \to \mathcal{B}(\mathcal{H})$ we can define a family of maps $\pi_t = Ad_{s^t} \circ \pi$ such that*

$$\|\pi_t\| \le \|\pi\|^{1-t} \text{ and } \|\pi_t\|_{c.b.} = \|\pi\|_{c.b.}^{1-t}.$$

The previous theorem was first obtained by Schlicht (see Lemma 3.6, Corollary 3.7 and Lemma 5.3 in [59]) without using explicitly the geometric properties of the function D_H and considering the case of a geodesic $\gamma_{1,s^2}(t) = s^{2t}$.

Remark 3.3.21. *If H is a unitarizable group then for $h \in H$ and a positive unitarizer s of H*

$$\|h\| = \|s(s^{-1}hs)s^{-1}\| \le \|s\|\|s^{-1}hs\|\|s^{-1}\| = \|s\|\|s^{-1}\|$$

since $s^{-1}hs$ is unitary. Taking the supremum over $h \in H$ and the infimum over positive unitarizers s we obtain

$$|H| \le Sim(H).$$

If we take logarithms we see that this inequality is equivalent to

$$D_H(1) \le 2dist(1, P^H).$$

This inequality can also be obtained using the fact that D_H is 2-Lipschitz. If $a \in P^H$ is such that $dist(1, P^H) = d(1, a)$, then since $D_H(a) = 0$

$$D_H(1) = |D_H(1) - D_H(a)| \le 2d(1, a) = 2dist(1, P^H).$$

Therefore, the fact that $|H| \le Sim(H)$ corresponds to the geometric fact that the diamater of the orbit of the identity element is less or equal than twice the distance between the identity element and the fixed point set of the action.

3.3.4 Minimality properties of canonical unitarizers

In [2] and [4] Andruchow, Corach and Stojanoff studied the differential geometry of spaces of representations of some classes of C^*-algebras and von Neumann algebras. Let A be a unital C^*-algebra and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on a separable Hilbert space $\mathcal{H}$. Denote by $R(A,\mathcal{B}(\mathcal{H}))$ the set of bounded unital homomorphisms from A to $\mathcal{B}(\mathcal{H})$ and by $R_0(A,\mathcal{B}(\mathcal{H}))$ the subset of $*$-representations. The group G of invertible operators in $\mathcal{B}(\mathcal{H})$ acts on $R(A,\mathcal{B}(\mathcal{H}))$ by inner automorphisms by the formula

$$(g\cdot\pi)(a)=(Ad_g\circ\pi)(a)=g\pi(a)g^{-1}$$

for $a\in A$ and $g\in G$. The group of unitary operators U acts on $R_0(A,\mathcal{B}(\mathcal{H}))$ in the same way. In this way $R(A,\mathcal{B}(\mathcal{H}))$ and $R_0(A,\mathcal{B}(\mathcal{H}))$ are homogeneous spaces. There is also an action of U on conditional expectations defined in $\mathcal{B}(\mathcal{H})$ given by $u\cdot E=Ad_u\circ E\circ Ad_{u^{-1}}$.

Given $\pi_0\in R_0(A,\mathcal{B}(\mathcal{H}))$ and a fixed conditional expectation $E_{\pi_0}:\mathcal{B}(\mathcal{H})\to\pi_0(A)'$ one obtains, by the splitting theorem of Porta and Recht in [57] (see Remark 2.2.19), that for every π in the G-orbit of π_0 in $R(A,\mathcal{B}(\mathcal{H}))$ there is a natural way of choosing a unique positive operator $s\in G$ such that $Ad_s\circ\pi$ is a $*$-representation in the following way: if $g\in G$ is such that $\pi_1=Ad_g\circ\pi_0$ the Porta-Recht splitting asserts that there are $u\in U$, $Y_0={Y_0}^*\in\pi_0(A)'$ and $Z_0={Z_0}^*\in Ker(E_{\pi_0})$ such that $g=ue^{Z_0}e^{Y_0}$. Then for $a\in A$

$$\begin{aligned}\pi_1(a)&=ue^{Z_0}e^{Y_0}\pi_0(a)e^{-Y_0}e^{-Z_0}u^*\\&=ue^{Z_0}\pi_0(a)e^{-Z_0}u^*\\&=e^{uZ_0u^*}u\pi_0(a)u^*e^{-uZ_0u^*}\\&=e^{Ad_uZ_0}(u\cdot\pi_0)(a)e^{Ad_uZ_0}.\end{aligned}$$

If we define $\rho=u\cdot\pi_0=Ad_u\circ\pi_0$, $X_0=Ad_u(Z_0)$ and $E_\rho=u\cdot E_{\pi_0}=Ad_u\circ E_{\pi_0}\circ Ad_{u^{-1}}$, then $Ad_{e^{-X_0}}\circ\pi_1=\rho\in R_0(A,\mathcal{B}(\mathcal{H}))$ and $X_0\in Ker(E_\rho)$. Also X_0 and ρ are unique with this properties: if $\rho'=v\cdot\pi_0$ for a unitary v and $X_0'\in Ker(E_{\rho'})$, where $E_{\rho'}=v\cdot E_{\pi_0}$, then $Ad_{e^{-X_0'}}\circ\pi_1=\rho'\in R_0(A,\mathcal{B}(\mathcal{H}))$ implies $X_0=X_0'$ and $\rho=\rho'$. See Remark 5.7. and Theorem 5.8 in [2] for further details.

Andruchow, Corach and Stojanoff asked before Remark 5.9 in [2] and in [4, Section 1.5] if e^{-X_0}, which is the canonical invertible operator such that $Ad_{e^{-X_0}}\circ\pi_1$ is a $*$-representation, satisfies $\|e^{X_0}\|\|e^{-X_0}\|=\|\pi_1\|_{c.b.}$. To give an answer and a geometrical insight to this question we recall a theorem proved by Conde and Laratonda in [16, Corollary 4.39] stated in the case of operator algebras and conditional expectations.

Theorem 3.3.22. *Let A be a C^*-algebra and B a C^*-subalgebra of A. Let $E : A \to B$ be a conditional expectation and let*

$$(A_s \cap Ker(E)) \times B_s \to P$$

$$(X, Y) \mapsto e^Y e^X e^Y$$

be the CPR splitting of P. Then $\|(I-E)|_{A_s}\| = 1$ if and only if for every $X \in A_s \cap Ker(E)$ and $Y \in B_s$ a closest point in $exp(B_s)$ to $e^Y e^X e^Y$ is e^{2Y}, i.e.

$$dist(exp(B_s), e^Y e^X e^Y) = d(e^{2Y}, e^Y e^X e^Y) = \|log(e^X)\| = \|X\|.$$

Theorem 3.3.23. *Assuming the notation and construction of canononical unitarizers of the beginning of this section*

$$\|\pi_1\|_{c.b.} = exp(dist(e^{-2X_0}, P^{\rho(U_A)})) = exp(dist(e^{-2X_0}, exp(\rho(U_A)' \cap \mathcal{B}(\mathcal{H})_s))).$$

If $\|I - E_{\pi_0}\| = 1$ then $\|e^{X_0}\|\|e^{-X_0}\| = \|\pi_1\|_{c.b.}$.

Proof. Note that

$$\begin{aligned}
\|\pi_1\|_{c.b.} &= Sim(\pi_1(U_A)) \\
&= exp(dist(1, P^{\pi_1(U_A)})) \text{ by Proposition 3.3.12} \\
&= exp(dist(1, P^{e^{X_0}\rho(U_A)e^{-X_0}})) \text{ since } Ad_{e^{-X_0}} \circ \pi_1 = \rho \\
&= exp(dist(1, e^{X_0} P^{\rho(U_A)} e^{X_0})) \text{ by Proposition 3.2.4} \\
&= exp(dist(1, I_{e^{X_0}}(P^{\rho(U_A)}))) = exp(dist(I_{e^{-X_0}}(1), P^{\rho(U_A)})) \\
&= exp(dist(e^{-2X_0}, P^{\rho(U_A)})) \\
&= exp(dist(e^{-2X_0}, exp(\rho(U_A)' \cap \mathcal{B}(\mathcal{H})_s))) \text{ by Remark 3.2.6 .}
\end{aligned}$$

This proves the first equality.

If $\|I - E_{\pi_0}\| = 1$, since

$$E_\rho : \mathcal{B}(\mathcal{H}) \to Ad_u(\pi_0(U_A)') = Ad_u(\pi_0(U_A))' = \rho(U_A)'$$

and

$$\|I - E_\rho\| = \|Ad_u \circ (I - E_{\pi_0}) \circ Ad_{u^{-1}}\| \leq \|I - E_{\pi_0}\| = 1,$$

we get $\|I - E_\rho\| = 1$. Therefore by Theorem 3.3.22

$$dist(exp(\rho(U_A)' \cap \mathcal{B}(\mathcal{H})_s), e^X) = d(1, e^X) = \|X\|$$

for $X \in Ker(E_\rho)$. Hence, since $X_0 \in Ker(E_\rho)$

$$\|\pi_1\|_{c.b.} = exp(dist(e^{-2X_0}, exp(\rho(U_A)' \cap \mathcal{B}(\mathcal{H})_s))) = e^{\|2X_0\|}.$$

Since e^{-X_0} is an orthogonalizer of π_1 we get $\|\pi_1\|_{c.b.} \leq \|e^{X_0}\|\|e^{-X_0}\|$. Also $\|e^{X_0}\|\|e^{-X_0}\| \leq e^{2\|X_0\|}$ always holds, hence we get the equality stated in the theorem. □

We next give an example of a conditional expectation E satisfying $\|I - E\| = 1$.

Example 3.3.24. *Let $A = \mathcal{B}(\mathcal{H})$ be the C^*-algebra of bounded operators acting on a Hilbert space $\mathcal{H}$. Let $p \in \mathcal{B}(\mathcal{H})$ be an orthogonal proyection, so that $q = 2p - 1$ is a self-adjoint unitary, i.e. a symmetry. Then*

$$E : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad X \mapsto \frac{1}{2}(X + qXq) = pXp + (1-p)X(1-p)$$

is a conditional expectation onto the subalgebra $A = \{X \in \mathcal{B}(\mathcal{H}) : pX = Xp\}$. Since

$$\|X - E(X)\| = \|X - \frac{1}{2}(X + qXq)\| = \|\frac{1}{2}(X - qXq)\| \leq \|X\|$$

we conclude that $\|I - E\| = 1$. If $\mathcal{H}_1$ is the range of p and $\mathcal{H}_2$ is the range of $1-p$ then we have the orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. If we write the operators in $\mathcal{B}(\mathcal{H})$ as 2×2 matrices with respect to this decomposition then

$$E : \mathcal{B}(\mathcal{H}) \to A, \qquad \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \mapsto \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}.$$

3.4 Groups of isometries of CAT(0) spaces in the context of operator algebras

If G is an amenable group with invariant mean ϕ and $\pi : G \to \mathcal{B}(\mathcal{H})$ is a uniformly bounded representation, then using abusive notation

$$s = \left(\int_G \pi(g)\pi(g)^* d\phi(g) \right)^{\frac{1}{2}}$$

is a unitarizer of $H = \pi(G)$, so that the unitarizer is the square root of the center of mass of $\{hh^*\}_{h \in H}$, see [25, 26, 47]. In the opposite direction, Ehrenpreis and Mautner [29] constructed a nonunitarizable bounded representation of $SL_2(\mathbb{R})$ on $\mathcal{H}$, and the group $SL_2(\mathbb{R})$ was later replaced by any countable group containing the free group $\mathbb{F}_2$.

We present a proof of the fact that a uniformly bounded group H of invertible elements is similar to a unitary group in two cases:

- The group H lies in a finite von Neumann algebra.
- The group H is close to the trivial group in that $sup_{h\in H}\|h-1\|_2 < \infty$, where $\|\cdot\|_2$ is the Hilbert-Schmidt norm.

The proofs involve new metric geometric arguments in the non-positively curved space of positive invertible operators of the algebra which yield an explicit unitarizer. In these contexts where the metric is derived from a Hilbertian norm the Bruhat-Tits fixed point theorem implies that the square root of the circumcenter of $\{hh^*\}_{h\in H}$ is a unitarizer of H.

In 1974 Vasilescu and Zsido proved the unitarizability of uniformly bounded groups of invertible operators in finite von Neumann algebras using the Ryll-Nardzewsky fixed point theorem [64] and the weak topologies of the operator algebra.

3.4.1 CAT(0) spaces and groups of isometries

We recall some well-known results from metric geometry. A general reference is [13]. For the convenience of the the reader we include the proof of the Bruhat-Tits fixed point theorem.

Definition 3.4.1. *A metric space (X, d) satisfies the **semi-parallelogram law** if for all $x, y \in X$ there is a $w \in X$ such that for all $z \in X$ the following inequality holds*

$$d(x,y)^2 + 4d(w,z)^2 \leq 2[d(x,z)^2 + d(y,z)^2].$$

*A **CAT(0) space** or Bruhat-Tits space is a complete metric space in which the semi-parallelogram law holds.*

If we set $z = x$ and $z = y$ in the semi-parallelogram law it follows that

$$2d(w,x) = 2d(w,y) \leq d(x,y).$$

Using the triangle inequality we see that

$$d(x,y) \leq d(w,x) + d(w,y) \leq \frac{1}{2}d(x,y) + \frac{1}{2}d(x,y) = d(x,y)$$

so that

$$d(z,x) = d(z,y) = \frac{1}{2}d(x,y).$$

The point w is uniquely determined because if w' is another such point, we put $z = w'$ in the law to get

$$d(x,y)^2 + 4d(w,w')^2 \leq 2[d(x,w')^2 + d(y,w')^2] \leq 2[(\frac{1}{2}d(x,y))^2 + (\frac{1}{2}d(x,y))^2] = d(x,y)$$

so that $d(w,w') = 0$.

Remark 3.4.2. *The unique point z in the definition of semi-paralleogram law is called the **midpoint** between x and y and we denote it by $m(x, y)$. We therefore have a function $m : X \times X \to X$ called the midpoint map.*

The following result is Serre's Lemma [36, Ch. XI, Lemma 3.1].

Lemma 3.4.3. *Let (X, d) be a CAT(0) space and S a bounded subset of X. Then there is a unique closed ball $B_r[x]$ of minimal radius containing S.*

Proof. To prove uniqueness suppose there are two balls $B_r[x]$ and $B_r[y]$ of minimal radius containing S. Let $z \in S$ so $d(z, x) \leq r$ and $d(z, y) \leq r$. Let w be the midpoint between x and y. By the semi-parallelogram law

$$d(x, y)^2 + 4d(w, z)^2 \leq 2[d(x, z)^2 + d(y, z)^2] \leq 4r^2$$

and therefore

$$d(x, y)^2 \leq 4(r^2 - d(w, z)^2).$$

For each $\epsilon > 0$ there is a $z \in S$ such that $d(z, w) \geq r - \epsilon$ since otherwise there is an $\epsilon > 0$ such that $d(z, w) < r - \epsilon$ for all $z \in S$ so that $S \subseteq B_{r-\epsilon}[w]$ contradicting the minimality of the balls $B_r[x]$ and $B_r[y]$. It follows that $d(x, y) = 0$, that is $x = y$.

To prove existence, let $(x_n)_n$ be a sequence of points which are centers of balls of radius r_n which contain S, with

$$r_n \to r_0 = inf\{r : \text{ there is } x \in X \text{ such that } S \subseteq B_r[x]\}.$$

If the sequence $(x_n)_n$ is a Cauchy sequence, then it convergerges to a point x_0 and since $S \subseteq B_{r_n}[x_n]$ for all $n \in \mathbb{N}$ we see that $B_{r_o}[x_0]$ is the unique closed ball of minimum radius containing S.

Let w_{mn} be the midpoint between x_n and x_m. By the minimality of r_0, given $\epsilon > 0$ it follows that $S \nsubseteq B_{r_0-\epsilon}[w_{mn}]$ so there is an $x \in S$ such that

$$d(x, w_{mn})^2 \geq r^2 - \epsilon.$$

We apply the semi-paralleogram law. Then

$$d(x_m, x_n)^2 + 4d(w_{mn}, x)^2 \leq 2[d(x_m, x)^2 + d(x_n, x)^2]$$

so that

$$\begin{aligned} d(x_m, x_n)^2 &\leq 2[d(x_m, x)^2 + d(x_n, x)^2] - 4d(w_{mn}, x)^2 \\ &\leq 2[d(x_m, x)^2 + d(x_n, x)^2] - 4r^2 + 4\epsilon \\ &\leq 2r_m^2 + 2r_n^2 - 4r^2 + 4\epsilon \end{aligned}$$

thus proving that $(x_n)_n$ is Cauchy. ☐

Definition 3.4.4. *The center y of the closed ball $B_r[y]$ in the previous lemma is called the* ***circumcenter*** *of the bounded set S.*

Using Serre's lemma one can prove the Bruhat-Tits fixed point theorem, [12].

Theorem 3.4.5. *If (X,d) is a CAT(0) space and $I : G \to Isom(X)$ is an action of a group G on X by isometries which has a bounded orbit, then the circumcenter of each orbit is a fixed point of the action.*

Proof. We denote the action by $g \cdot x$ for $g \in G$ and $x \in X$. Since the action is isometric and there is a bounded orbit all orbits are bounded. For $x \in X$ let $B_r[y]$ be the unique closed ball of minimal radius which contains $G \cdot x$. The existence of this ball is given by Lemma 3.4.3. If $g \in G$ then $G \cdot x = g \cdot (G \cdot x) \subseteq g \cdot B_r[y] = B_r[g \cdot y]$ where the last equality follows since the action is isometric. From the uniqueness of the closed balls of minimal radius containing $G \cdot x$ we conclude that $g \cdot y = y$. Therefore, $g \cdot y = y$ for every $g \in G$ and y is a fixed point of the action. □

3.4.2 Finite von Neumann algebras

The metric geometry of the cone of positive invertible operators in a finite von Neumann algebra was studied in [5, 16]. In this section we recall some facts from these papers.

Let A be a von Neumann algebra with a finite (normal, faithful) trace τ. Denote by A_s the set of self-adjoint operators of A, by G the group of invertible operators, by U the group of unitary operators, and by P the set of positive invertible operators

$$P = e^{A_s} = \{a \in G : a > 0\}.$$

Since P is an open subset of A_s in the norm topology it is a submanifold of A_s and its tangent spaces will be identified with A_s endowed with the uniform norm $\|\cdot\|$.

We make of P a weak Riemann-Finsler manifold by assigning for each $a \in P$ the following 2-norm to the tangent space $T_a(P) \simeq A_s$

$$\|X\|_{a,2} = \|a^{-\frac{1}{2}} X a^{-\frac{1}{2}}\|_2, \qquad \text{for } X \in A_s \simeq T_a(P)$$

where

$$\|X\|_2 = \tau(X^2)^{\frac{1}{2}} \qquad \text{for } X \in A_s.$$

Note that $\|X\|_2 = \tau(X^2)^{\frac{1}{2}} \leq \|X\|$ for all $X \in A_s$. Since there is no $M > 0$ such that$\|X\|_2 = \tau(X^2)^{\frac{1}{2}} \geq M\|X\|$ for all $X \in A_s$ we see that this tangent norm is not compatible with the manifold structure of P in the sense of Definition 1.5.1.

One obtains a geodesic distance d on P by considering for $a, b \in P$

$$d(a,b) = inf\{Length(\gamma) : \gamma \text{ is a piecewise smooth curve joining } a \text{ and } b\},$$

where smooth means differentiable in the norm induced topology and the lenght of a curve $\gamma : [0,1] \to P$ is measured using the norm above:

$$Length(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t),2} dt.$$

If $\mathcal{A}$ is finite dimensional, i.e. a sum of matrix spaces, this metric is well-known: it is the non positively curved Riemannian metric on the set of positive definite matrices [46].

If $\mathcal{A}$ is of type II_1, the trace inner product is not complete, so that P is not a Hilbert-Riemann manifold and (P, d) is not a complete metric space, see Remark 3.21 in [16].

By [5, Theorem 3.1 and Remark 3.2] the unique minimizing geodesic between a and b for $a, b \in P$ is given by

$$\gamma_{a,b}(t) = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^t a^{\frac{1}{2}}$$

and has lenght equal to

$$d(a,b) = Length(\gamma_{a,b}) = \|log(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})\|_2.$$

The action of G on P given by $I_g(a) = gag^*$ is isometric, i.e. $d(I_g(a), I_g(b)) = d(a,b)$, and sends geodesic segments to geodesic segments, i.e. $I_g \circ \gamma_{a,b} = \gamma_{I_g(a), I_g(b)}$ for all $a, b \in P$ and $g \in G$. See the Introduction of [5].

By [5, Lemma 3.5] P has the exponential metric increasing property, i.e. for $X, Y \in A_s \simeq T_1 P$

$$\|X - Y\|_2 \leq d(exp_1(X), exp_1(Y))$$

and

$$\|X\|_2 = d(exp_1(X), exp_1(0)) = d(exp_1(X), 1).$$

Proposition 3.4.6. *Let $a \in P$ and $\gamma : [0,1] \to P$ be a geodesic. Then*

$$d(\gamma(0), \gamma(1))^2 + 4d(a, \gamma(\frac{1}{2}))^2 \leq 2(d(a, \gamma(0))^2 + d(a, \gamma(1))^2)$$

so the metric space (P, d) satisfies the semi-parallelogram law, see Definition 3.4.1 above.

Proof. Since the action I is transitive and sends geodesic segments to geodesic segments we can assume that $\gamma(\frac{1}{2}) = 1$. Therefore there are $X, Y \in A_s \simeq T_1 P$ such that

$$exp_1(-X) = \gamma(0), \quad exp_1(X) = \gamma(1), \quad exp_1(Y) = a.$$

Since the parallelogram law holds in T_1P we have

$$2\|Y\|_2^2 + 2\|X\|_2^2 = \|Y - X\|_2^2 + \|Y + X\|_2^2,$$

which multiplied by two yields

$$4\|Y\|_2^2 + \|X - (-X)\|_2^2 = 2\|Y - X\|_2^2 + 2\|Y - (-X)\|_2^2.$$

Using the exponential metric increasing property

$$\begin{aligned} 4d(exp_1(Y), 1)^2 &+ d(exp_1(X), exp_1(-X))^2 \\ &\leq 2d(exp_1(Y), exp_1(X))^2 + 2d(exp_1(Y), exp_1(-X))^2. \end{aligned}$$

Substituing we get

$$4d(a, \gamma(\frac{1}{2}))^2 + d(\gamma(0), \gamma(1))^2 \leq 2(d(a, \gamma(1))^2 + d(a, \gamma(0))^2).$$

□

Existence of unitarizers of bounded subgroups

A subset $C \subseteq P$ is *geodesically convex* if $\gamma_{a,b}(t) \in C$ for every $a, b \in C$ and $t \in [0, 1]$, and is *midpoint convex* if $\gamma_{a,b}(\frac{1}{2}) \in C$ for every $a, b \in C$. Note that a geodesically convex set is midpoint convex.

Lemma 3.4.7. *If $C \subseteq P$ is geodesically convex then its closure $\overline{C}$ in (P, d) is geodesically convex.*

Proof. By [5, Cor. 3.4] the distance along two geodesics is convex, i.e.

$$[0, 1] \to [0, +\infty) \quad t \mapsto d(\gamma_{a_1,b_1}(t), \gamma_{a_2,b_2}(t))$$

is convex for all $a_1, b_1, a_2, b_2 \in P$. Hence, for $t \in [0, 1]$ fixed, $(a, b) \mapsto \gamma_{a,b}(t)$ is d-continuous.

If $a, b \in \overline{C}$ and $t \in [0, 1]$ let $(a_n)_n, (b_n)_n$ be sequences in C such that $a_n \to a$ and $b_n \to b$. Since C is geodesically convex $\gamma_{a_n,b_n}(t) \in C$ for all $n \in \mathbb{N}$. The d-continuity of $(a, b) \mapsto \gamma_{a,b}(t)$ implies that $\gamma_{a_n,b_n}(t) \to \gamma_{a,b}(t)$, so that $\gamma_{a,b}(t) \in \overline{C}$. □

Lemma 3.4.8. *For $0 < c_1 < c_2$ the interval $P_{c_1,c_2} = \{a \in P : c_1 1 \leq a \leq c_2 1\}$ is geodesically convex.*

Proof. By Proposition 4.2.8 and Excercise 4.6.46 in [35] if $t \in (0,1]$ and a and b are positive invertible elements in a C^*-algebra A such that $a \leq b$ then $a^t \leq b^t$ (this is the Loewner-Heinz inequality). It is easily verified that for $c \in A$ if $a \leq b$ then $cac^* \leq cbc^*$. Therefore, if $a, b \in P$ such that

$$c_1 1 \leq a, b \leq c_2 1.$$

then

$$c_1 a^{-1} \leq a^{-\frac{1}{2}} b a^{-\frac{1}{2}} \leq c_2 a^{-1}$$

and exponentiating with $t \in [0,1]$ we get

$$c_1^t a^{-t} \leq (a^{-\frac{1}{2}} b a^{-\frac{1}{2}})^t \leq c_2^t a^{-t}.$$

We conclude that

$$c_1 1 \leq c_1^t c_1^{1-t} 1 \leq c_1^t a^{1-t} \leq a^{\frac{1}{2}} (a^{-\frac{1}{2}} b a^{-\frac{1}{2}})^t a^{\frac{1}{2}} \leq c_2^t a^{1-t} \leq c_2^t c_2^{1-t} 1 \leq c_2 1.$$

□

Lemma 3.4.9. *For* $0 < c_1 < c_2$ *the interval* $P_{c_1,c_2} = \{a \in P : c_1 1 \leq a \leq c_2 1\}$ *endowed with the metric* d *is a complete and bounded metric space.*

Proof. In P_{c_1,c_2} the linear metric and the rectifiable distance are equivalent [16, Prop. 3.2], i.e. there are $C, C' > 0$ such that $\|a-b\|_2 \leq C d(a,b)$ and $d(a,b) \leq C'\|a-b\|_2$ for all $a, b \in P_{c_1,c_2}$.

Since $\|\cdot\|_2$ induces a complete metric on subsets of A which are weakly closed and bounded in the uniform norm, and P_{c_1,c_2} is weakly closed and bounded in the uniform norm, we conclude that (P_{c_1,c_2}, d) is a complete metric space.

Also, (P_{c_1,c_2}, d) is a bounded metric space because $d(a,b) \leq C'\|a-b\|_2 \leq C'\|a-b\| \leq 2C'c_2$ for all $a, b \in P_{c_1,c_2}$. □

Theorem 3.4.10. *If* $H \subseteq G$ *is a subgroup such that* $\sup_{h \in H} \|h\| = |H| < \infty$ *then there is an* $s \in P_{|H|^{-1},|H|}$ *such that* $s^{-1} H s \subseteq U$.

Proof. Consider the isometric action $I : H \to Isom(P)$ given by $I_h(a) = hah^*$ for $h \in H$ and $a \in P$. We denote the action by $h \cdot a = I_h(a)$. Take $X_1 = H \cdot 1$ and define inductively $X_{n+1} = \{\gamma_{a,b}(t) : a, b \in X_n, t \in [0,1]\}$ for $n \geq 1$. Let

$$conv(H \cdot 1) = \bigcup_{n \in \mathbb{N}} X_n.$$

Since $P_{|H|^{-2},|H|^2}$ is geodesically convex and the action sends geodesic segments to geodesic segments, if $X_n \subseteq P_{|H|^{-2},|H|^2}$ then $X_{n+1} \subseteq P_{|H|^{-2},|H|^2}$ for all $n \in \mathbb{N}$. Therefore $conv(H \cdot 1) \subseteq P_{|H|^{-2},|H|^2}$ follows from $X_1 = H \cdot 1 = \{hh^*\}_{h\in H} \subseteq P_{|H|^{-2},|H|^2}$. Also, using the fact that $P_{|H|^{-2},|H|^2}$ is closed in (P, d) we conclude that $\overline{conv}(H \cdot 1) \subseteq P_{|H|^{-2},|H|^2}$.

Since the action sends geodesic segments to geodesic segments, if X_n is invariant under the action I then X_{n+1} is invariant for all $n \in \mathbb{N}$. Since $X_1 = H \cdot 1$ is invariant, we conclude that $conv(H \cdot 1)$ is invariant. The action is also isometric, hence $\overline{conv}(H \cdot 1)$ is invariant and we can restrict the action I to this subset.

The space $(\overline{conv}(H \cdot 1), d)$ is midpoint convex and the semi-parallelogram law holds in P, hence this law also holds in $(\overline{conv}(H \cdot 1), d)$. Since $\overline{conv}(H \cdot 1)$ is a closed subset of the complete metric space $(P_{|H|^{-2},|H|^2}, d)$ the space $(\overline{conv}(H \cdot 1), d)$ is complete. We conclude that $(\overline{conv}(H \cdot 1), d)$ is a complete metric space in which the semi-parallelogram holds.

Since $(P_{|H|^{-2},|H|^2}, d)$ is a bounded metric space $\overline{conv}(H \cdot 1)$ is a bounded set. Therefore the restricted action has bounded orbits, and Theorem 3.4.5 states that there is an $a \in \overline{conv}(H \cdot 1)$ such that $I_h(a) = hah^* = a$ for all $h \in H$. Therefore by Proposition 3.2.3 $a^{-\frac{1}{2}} H a^{\frac{1}{2}} \subseteq U$, i.e. $s = a^{\frac{1}{2}}$ is a unitarizer of H.

Because the square root is an operator monotone function and $a \in P_{|H|^{-2},|H|^2}$, we see that $s = a^{\frac{1}{2}} \in P_{|H|^{-1},|H|}$. □

This last result was published in [43].

Invariants given by a conditional expectation

We want to further analyze the orbit structure of the action I. Using Proposition 3.2.4 we can assume that H is a group of unitaries. By a theorem of Takesaki [61] there is a conditional expectation $E : A \to H' \cap A$ compatible with the trace, i.e. $E(\tau(x)) = E(x)$ for $\in A$. The conditional expectation provides an orthogonal splitting

$$A = (A \cap H') \oplus_\tau Ker(E)$$

with respect to the inner product $\langle x, y\rangle = \tau(y^* x)$. Theorem 5.4 and Corollary 5.5 in [5] in this case imply the following result.

Proposition 3.4.11. *Assuming the context of this section, let*

$$(A_s \cap Ker(E)) \times (A_s \cap H') \to P$$

$$(X, Y) \mapsto e^Y e^X e^Y$$

be the bijection given by the Porta-Recht splitting. If $a = e^Y e^X e^Y$ is the factorization of $a \in P$ then the closest point in $exp(A_s \cap H')$ to a is e^{2Y}, and this point is unique with this property.

Proposition 3.4.12. *The sets $e^Y e^{(A_s \cap Ker(E))} e^Y$ are invariant for the action I. The circumcenter of any orbit in $e^Y e^{(A_s \cap KerE)} e^Y$ is e^{2Y}.*

Proof. We have

$$Ker(E) = (A \cap H')^{\perp} = \{X \in A : \tau(XY^*) = 0 \text{ for all } Y \in H' \cap A\}.$$

Then $Ker(E)$ is Ad_H-invariant because if $X \in Ker(E)$, $Y \in A \cap H'$ and $h \in H$ then

$$\tau(Ad_h(X)Y^*) = \tau(hXh^{-1}Y^*) = \tau(hXY^*h^{-1}) = \tau(h^{-1}hXY^*) = \tau(XY^*).$$

If $a = e^Y e^X e^Y$ is a decomposition of a given by the Porta-Recht splitting then

$$I_h(a) = hah^{-1} = he^Y e^X e^Y h^{-1} = e^{Ad_h(Y)} e^{Ad_h(X)} e^{Ad_h(Y)} = e^Y e^{Ad_h(X)} e^Y$$

so that the sets $e^Y e^{(A_s \cap KerE)} e^Y$ are invariant for the action I.

An orbit in $e^Y e^{(A_s \cap KerE)} e^Y$ is of the form $\{e^Y e^{Ad_h(X)} e^Y : h \in H\}$ for some $X \in A_s \cap Ker(E)$, and its circumcenter is a fixed point of the action which is closest to each element in the orbit by Lemma 3.4.3 and Theorem 3.4.5. This point is e^{2Y} by Proposition 3.4.11.

□

3.4.3 Hilbert-Schmidt perturbations of the identity

We next consider the case when the group is close in some sense to the trivial group $\mathbb{T}1$.

The geometry of the positive invertible unitized Hilbert-Schmidt operators was studied in [37]. Let $\mathcal{B}_2(\mathcal{H})$ stand for the bilateral ideal of Hilbert-Schmdit operators of $\mathcal{B}(\mathcal{H})$. Recall that $\mathcal{B}_2(\mathcal{H})$ is a Banach algebra without unit when given the *Hilbert-Schmidt norm* $\|a\|_2 = tr(a^*a)^{\frac{1}{2}}$ and that $\|a\| \leq \|a\|_2$ for $a \in \mathcal{B}_2(\mathcal{H})$. We consider the following complex linear subalgebra of $\mathcal{B}(\mathcal{H})$

$$\mathcal{B}_2(\mathcal{H}) + \mathbb{C}1 = \{a + \lambda 1 : a \in \mathcal{B}_2(\mathcal{H}), \lambda \in \mathbb{C}\}$$

consisting of the Hilbert-Schmidt perturbations of scalar multiples of the identity. There is a natural Hilbert space structure for this subspace, where the scalar operators are orthogonal to Hilbert-Schmidt operators, which is given by the inner product

$$\langle a + \lambda, b + \beta \rangle_2 = tr(ab^*) + \lambda\bar{\beta}.$$

This product is well defined and positive definite. We see that $\mathcal{B}_2(\mathcal{H}) + \mathbb{C}1$ is complete with this norm since the Hilbert-Schmidt inner product induces a complete norm on the ideal of Hilbert-Schmidt operators (see Theorem 3.4.9 in [52]). The model space we are considering is the real space of $\mathcal{B}_2(\mathcal{H}) + \mathbb{C}1$, which is $\mathcal{B}_2(\mathcal{H})_s + \mathbb{R}1$. This space inherits the structure of a real Banach space, and with the same inner product it becomes a real Hilbert space. Inside $\mathcal{B}_2(\mathcal{H})_s + \mathbb{R}1$ consider the open subset of positive invertible operators

$$P = \{a : a \in \mathcal{B}_2(\mathcal{H})_s + \mathbb{R}1, a > 0\}.$$

For $a \in P$ we identify T_aP with $\mathcal{B}_2(\mathcal{H})_s + \mathbb{R}1$ and endow this manifold with a real Riemannian metric by means of the formula

$$\langle X, Y\rangle_a = \langle a^{-1}X, a^{-1}Y\rangle_2.$$

The unique geodesic $\gamma_{a,b} : [0, 1] \to P$ joining a and b is given by

$$\gamma_{a,b}(t) = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^t a^{\frac{1}{2}}$$

and realizes the distance, i.e.

$$d(a, b) = Length(\gamma_{a,b}) = \|log(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})\|_2,$$

see Theorem 3.8 and Remark 3.9 in [37].

With this metric P is a complete metric space. By Remark 2.4 in [37] the exponential map of P at $a \in P$ is given by $exp_a : T_aP \simeq \mathcal{B}_2(\mathcal{H})_s + \mathbb{R}1 \to P$

$$exp_a(X) = a^{\frac{1}{2}}e^{a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}}a^{\frac{1}{2}} = ae^{a^{-1}X} = e^{Xa^{-1}}a \text{ for } X \in \mathcal{B}_2(\mathcal{H})_s + \mathbb{R}1 \simeq T_aP.$$

The metric in P is invariant for the action of the group of invertible elements: if g is an invertible operator in $\mathcal{B}_2(\mathcal{H}) + \mathbb{C}1$ then $I_g : P \to P$ is an isometry, see Lemma 2.5 in [37].

By Lemma 3.11 in [37] the exponential map increases distance and preserves distance of geodesic rays, i.e.

$$\|X - Y\|_2 \leq d(e^X, e^Y) \text{ and } \|X\|_2 = d(1, e^X).$$

From this and the invariance of the distance under the action I it follows in the same way as in Proposition 3.4.6 that the semi-parallelogram law holds in P.

Theorem 3.4.13. *If H is a group of invertible elements in $\mathcal{B}_2(\mathcal{H})+\mathbb{C}1$ such that $sup_{h\in H}\|hh^* - 1\|_2 = C < \infty$ then there is an s in P such that $s^{-1}Hs$ is a group of unitaries.*

Proof. Since $sup_{h\in H}\|hh^* - 1\|_2 = C < \infty$, for $h \in H$

$$\|hh^*\| - \|1\| \leq \|hh^* - 1\| \leq \|hh^* - 1\|_2 \leq C,$$

so that

$$(C+1)^{-1}1 \leq hh^* \leq (C+1)1.$$

We want to prove that $diam(\mathcal{O}_H(1)) = sup_{h\in H}\|log(hh^*)\|_2$ is finite. For $h \in H$, since $hh^* - 1$ is compact hh^* is diagonalizable and has eigenvalues $(s_j)_j \subseteq [(C+1)^{-1}, (C+1)]$. Then

$$\|hh^* - 1\|_2^2 = \sum_j (s_j - 1)^2 \leq C^2.$$

We see that $log(hh^*)$ is diagonalizable and has eigenvalues $(log(s_j))_j$. Let D be a real number such that $|log(x)| \leq D|x-1|$ for all $x \in [(C+1)^{-1}, (C+1)]$. Then

$$\|log(hh^*)\|_2^2 = \sum_j log(s_j)^2 \leq \sum D^2(s_j - 1)^2 \leq D^2C^2.$$

Since the last inequality holds for all $h \in H$ we see that $diam(\mathcal{O}_H(1)) \leq D^2C^2$. Since $\mathcal{O}_H(1)$ is bounded, by Theorem 3.4.5 the circumcenter $a \in P$ of this set is a fixed point for the action I restricted to H. By Proposition 3.2.3 $s = a^{\frac{1}{2}}$ is a unitarizer of H. □

Proposition 3.4.14. *If H is a group of invertible elements such that $sup_{h\in H}\|h - 1\|_2 = M < \infty$, then H is a group of invertible elements in $B_2(H) + \mathbb{C}1$ such that $sup_{h\in H}\|hh^* - 1\|_2 < \infty$ and is therefore unitarizable.*

Proof. That $H \subseteq \mathcal{B}_2(\mathcal{H}) + \mathbb{C}1$ is apparent. Since $\|h\| - \|1\| \leq \|h - 1\| \leq \|h - 1\|_2 \leq M$ for all $h \in H$ we see that $\|h\| \leq M + 1$ for all $h \in H$. Since

$$hh^* - 1 = hh^* - h + h - 1 = h(h^* - 1) + h - 1$$

for all $h \in H$ it follows that

$$\|hh^* - 1\|_2 \leq \|h\|\|h^* - 1\|_2 + \|h - 1\|_2 \leq (M+1)M + M$$

for all $h \in H$ so that $sup_{h\in H}\|hh^* - 1\|_2 = (M+1)M + M < \infty$. By Theorem 3.4.13 H is unitarizable. □

Bibliography

[1] R. Abraham J. E. Marsden, T. Ratiu, *Manifolds, tensor analysis, and applications.* Second edition. Applied Mathematical Sciences, 75. Springer-Verlag, New York, 1988

[2] E. Andruchow, G. Corach, D. Stojanoff, *A geometric characterization of nuclearity and injectivity.* J. Funct. Anal. 133 (1995), no. 2, 474–494.

[3] E. Andruchow, G. Corach, D. Stojanoff, *Geometrical Significance of Löwner-Heinz inequality.* Proc. Amer. Math. Soc. 128 (2000), no. 4, 1031-1037.

[4] E. Andruchow, G. Corach, D. Stojanoff, *The homogeneous space of representations of a nuclear C^*-algebra.* Harmonic analysis and operator theory (Caracas, 1994), 37–53, Contemp. Math., 189, Amer. Math. Soc., Providence, RI, 1995.

[5] E. Andruchow, G. Larotonda, *Nonpositively Curved Metric in the Positive Cone of a Finite von Neumann Algebra*, J. London Math. Soc. (2) 74 (2006), no. 1, 205-218.

[6] D. Beltiţă, J.E. Galé, *Holomorphic geometric models for representations of C^*-algebras.* Journal of Functional Analysis 255 (2008), no. 10, 2888–2932.

[7] D. Beltiţă, J.E. Galé, *Linear connections for reproducing kernels on vector bundles.* Mathematische Zeitschrift 277 (2014), no. 1-2, 29–62.

[8] D. Beltiţă, T. S. Ratiu, *Symplectic leaves in real Banach Lie-Poisson spaces.* Geom. Funct. Anal. 15 (2005), no. 4, 753–779.

[9] D. Beltiţă, *Smooth Homogeneous Structures in Operator Theory.* Chapman Hall CRC Monographs and Surveys in Pure and Applied Mathematics, 137. Chapman and Hall CRC, Boca Raton, FL, 2006.

[10] R. Bielawski, *Prescribing Ricci curvature on complexified symmetric spaces.* Math. Res. Lett. 11 (2004), 435–441.

[11] N. P. Brown, N. Ozawa, *C^*-algebras and finite-dimensional approximations.* Graduate Studies in Mathematics, 88. American Mathematical Society, Providence, RI, 2008.

[12] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local, I. Données radicielles valuées.* Inst. Hautes Études Sci. Publ. Math. 41 (1972), 5-252.

[13] D. Burago, Y. Burago, S. Ivanov, *A Course in Metric Geometry.* Amer. Math. Soc., Providence, 2001.

[14] E. Chiumiento, *Geometry of I-Stiefel manifolds.* Proc. Amer. Math. Soc. 138 (2010), no. 1, 341–353.

[15] C. Conde, G. Larotonda, *Manifolds of semi-negative curvature.* Proc. Lond. Math. Soc. (3) 100 (2010), no. 3, 670–704.

[16] C. Conde, G. Larotonda, *Spaces of nonpositive curvature arising from a finite algebra*, J. Math. Anal. Appl. 368 (2010), no. 2, 636–649.

[17] J. B. Conway, *A course in operator theory.* Graduate Studies in Mathematics, 21. American Mathematical Society, Providence, RI, 2000.

[18] G. Corach, J. E. Galé, *On amenability and geometry of spaces of bounded representations.* J. London Math. Soc. (2) 59 (1999), no. 1, 311–329.

[19] G. Corach, J. E. Galé, *Averaging with virtual diagonals and geometry of representations.* Banach algebras '97 (Blaubeuren), 87–100, de Gruyter, Berlin, 1998.

[20] G. Corach, H. Porta, L. Recht, *A geometric interpretation of Segal's inequality* $\|e^{X+Y}\| \leq \|e^{X/2}e^{Y}e^{X/2}\|$. Proc. Amer. Math. Soc. 115 (1992), no. 1, 229–231.

[21] G. Corach, H. Porta, L. Recht, *Geodesics and operator means in the space of positive operators.* Internat. J. Math. 4 (1993), no. 2, 193–202.

[22] G. Corach, H. Porta, L. Recht, *The geometry of the space of selfadjoint invertible elements in a C^*-algebra.* Integral Equations Operator Theory 16 (1993), no. 3, 333–359.

[23] G. Corach, H. Porta, L. Recht, *Convexity of the geodesic distance on spaces of positive operators.* Illinois J. Math. 38 (1994), no. 1, 87–94.

[24] K. R. Davidson, *C^*-algebras by example.* Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, 1996.

[25] M. M. Day, *Means for the bounded functions and ergodicity of the bounded representations of semi-groups.* Trans. Amer. Math. Soc. 69 (1950), 276-291.

[26] J. Dixmier, *Les moyennes invariantes dans les semi-groupes et leurs applications.* Acta Sci. Math. (Szeged) 12 (1950), 213-227.

[27] C. E. Durán, L. E. Mata-Lorenzo, L. Recht, *Metric geometry in homogeneous spaces of the unitary group of a C^*-algebra. I. Minimal curves.* Adv. Math. 184 (2004), no. 2, 342–366.

[28] C. E. Durán, L. E. Mata-Lorenzo, L. Recht, *Metric geometry in homogeneous spaces of the unitary group of a C^*-algebra. II. Geodesics joining fixed endpoints.* Integral Equations Operator Theory 53 (2005), no. 1, 33–50.

[29] L. Ehrenpreis, F. I. Mautner, *Uniformly bounded representations of groups.* Proc. Natl. Acad. Sci. USA 41 (1955), 231-233.

[30] J. E. Galé, *Geometría de órbitas de representaciones de grupos y álgebras promediables.* Revista de la Academia de Ciencias Exactas, Físicas, Químicas y Naturales de Zaragoza, ISSN 0370-3207, Nº. 61, 2006, pags. 7-46.

[31] W. M. Goldman, A. R. Magid (ed.), *Geometry of group representations.* Contemporary Math. 74, American Mathematical Society, Providence, RI, 1988.

[32] U. Haagerup, *Solution of the similarity problem for cyclic representations of C^*-algebras.* Ann. of Math. (2) 118 (1983), no. 2, 215–240.

[33] P. de la Harpe, *Classical Banach-Lie Algebras and Banach-Lie Groups of Operators in Hilbert Space.* Lecture Notes in Mathematics 285, Springer, Berlin, 1972.

[34] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces.* Corrected reprint of the 1978 original. Graduate Studies in Mathematics, 34. American Mathematical Society, Providence, RI, 2001

[35] R. V. Kadison, J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Volume I: Elementary Theory*, Amer. Math. Soc., Providence, 1997.

[36] S. Lang, *Fundamentals of Differential Geometry.* Graduate Texts in Mathematics, 191. Springer-Verlag, New York, 1999.

[37] G. Larotonda, *Nonpositive curvature: a geometrical approach to Hilbert-Schmidt operators.* Differential Geom. Appl. 25 (2007), no. 6, 679–700.

[38] J. Lawson, Y. Lim, *Symmetric spaces with convex metrics.* Forum Math. 19 (2007), no. 4, 571–602.

[39] L. Lempert, R. Szőke, *Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of Riemannian manifolds.* Math. Ann. 290 (1991), no. 4, 689–712.

[40] O. Loos, *Symmetric spaces. I: General theory.* W. A. Benjamin, Inc., New York-Amsterdam 1969

[41] L. E. Mata-Lorenzo, L. Recht, *Infinite-dimensional homogeneous reductive spaces.* Acta Cient. Venezolana 43 (1992), no. 2, 76–90.

[42] M. Miglioli, *Decompositions and complexifications of some infinite-dimensional homogeneous spaces.* J. Funct. Anal. 266 (2014), no. 11, 6599–6618.

[43] M. Miglioli, *Unitarization of uniformly bounded subgroups in finite von Neumann algebras.* Bull. London Math. Soc. (2014) 46 (6): 1264-1266.

[44] G. D. Mostow, *A structure theorem for homogeneous spaces.* Geom. Dedicata 114 (2005), 87-102.

[45] G. D. Mostow, *On covariant fiberings of Klein spaces.* Amer. J. Math. 77 (1955), 247-278.

[46] G. D. Mostow, *Some new decomposition theorems for semi-simple groups.* Mem. Amer. Math. Soc. (1955), no. 14, 31-54.

[47] M. Nakamura, Z. Takeda, *Group representation and Banach limit.* Tohoku Math. J. 3 (1951) 132-135.

[48] K.-H. Neeb, *A Cartan-Hadamard Theorem for Banach-Finsler manifolds.* Geom. Dedicata 95 (2002), 115-156.

[49] A. Odzijewicz, T.S. Ratiu, *Banach Lie-Poisson spaces and reduction.* Comm. Math. Phys. 243 (2003), no. 1, 1-54.

[50] M. I. Ostrovskii, V. S. Shulman, L. Turowska, *Unitarizable representations and fixed points of groups of biholomorphic transformations of operator balls.* J. Funct. Anal. 257 (2009), no. 8, 2476–2496.

[51] V. Paulsen, *Completely bounded maps and operator algebras.* Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, 2002.

[52] G. K. Pedersen, *Analysis now.* Graduate Texts in Mathematics, 118. Springer-Verlag, New York, 1989.

[53] G. Pisier, *A similarity degree characterization of nuclear C^*-algebras.* Publ. Res. Inst. Math. Sci. 42 (2006), no. 3, 691–704.

[54] G. Pisier, *Similarity problems and completely bounded maps.* Second, expanded edition. Includes the solution to "The Halmos problem". Lecture Notes in Mathematics, 1618. Springer-Verlag, Berlin, 2001.

[55] G. Pisier, *The similarity degree of an operator algebra.* Algebra i Analiz 10 (1998), no. 1, 132–186; translation in St. Petersburg Math. J. 10 (1999), no. 1, 103–146

[56] G. Pisier, *The similarity degree of an operator algebra.* II. Math. Z. 234 (2000), no. 1, 53–81.

[57] H. Porta, L. Recht, *Conditional expectations and operator decompositions.* Ann. Global Anal. Geom. 12 (4) (1994), 335–339.

[58] V. Runde, *Lectures on amenability.* Lecture Notes in Mathematics, 1774. Springer-Verlag, Berlin, 2002.

[59] P. Schlicht, *Amenable groups and a geometric view on unitarisability.* Phd Thesis, Universität Leipzig, (2013).

[60] B. Szőkefalvi-Nagy, *On uniformly bounded linear transformations in Hilbert space.* Acta Univ. Szeged. Sect. Sci. Math. 11 (1947)

[61] M. Takesaki, *Conditional expectations in von Neumann algebras.* J. Functional Analysis 9 (1972), 306–321.

[62] J. Tomiyama, *On the projection of norm one in W^*-algebras.* Proc. Japan Acad. 33 (1957), 608-612.

[63] H. Upmeier, *Symmetric Banach manifolds and Jordan C^*-algebras.* North-Holland Mathematics Studies, 104. Notas de Matemática [Mathematical Notes], 96. North-Holland Publishing Co., Amsterdam, 1985.

[64] F. H. Vasilescu, L. Zsido, *Uniformly bounded groups in finite W^*-algebras.* Acta Sci. Math. (Szeged), 36 (1974), 189-192.

Index